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Making Friends: the Role of Assortative Interests and Capacity Constraints

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Abstract

We investigate the properties of stability and efficiency of friendship networks when individuals (i) have arbitrary levels of assortative interests and (ii) are capacity constrained in their efforts to form connections. For intermediate levels of assortative interests, extreme forms of homophilic (or heterophilic) patterns may coexist as stable with more moderate homophilic patterns. In general, though, there is a natural positive relation between levels of assortative interests and homophily levels of stable patterns. For extreme forms of homophilic (resp., heterophilic) patterns to be stable, connections between agents with different (resp., same) characteristics must have a certain minimal (good) quality as well. The homophily features of stable networks are affected by the tightness of the capacity constraint and by the discrepancies between sizes of different-characteristic groups. Efficiency requires common aggregate qualities of connections (with respect to same-characteristic agents, on one side, and with respect to different-characteristic agents, on the other side) across all individuals within each different population group. Under very high (resp., low) levels of assortative interests, some particular forms of only extreme homophilic (resp., heterophilic) patterns are simultaneously stable and efficient. For intermediate levels of assortative interests, we identify a class of friendship networks that feature intermediate levels of homophily, and for which stability and efficiency are compatible.

Keywords: Friendship Networks, Assortative Interests, Homophily, Heterophily, Diversity, Integration.

Clasification: A12, A14, D01, D71, D85, J15, Z13.

Resumen

Estudiamos las propiedades de la estabilidad y eficiencia de redes de amistad cuando los individuos (i) tienen niveles arbitrarios de intereses asortativos y (ii) están restringidos en sus capacidades para formar conexiones. Para niveles intermedios de intereses asortativos, formas extremas de comportamientos homofílicos (o heterofílicos) pueden coexistir como estables con comportamientos homofílicos más moderados. En general, sin embargo, existe una relación positiva natural entre niveles de intereses asortativos y niveles de homofilia en redes estables. Para que redes con formas extremas de homofília (resp., heterofília) sean estables, las conexiones entre individuos con distintas (resp., idénticas) características deben tener una cierta calidad mínima también. Las propiedades de homofília de redes estables dependen de la severidad de las restricciones de capacidad y de las diferencias en tamaños entre grupos con diferentes características. Eficiencia requiere que las calidades de las conexiones agregadas (respecto a individuos de las mismas características, por un lado, y respecto a individuos con características distintas, por otro lado) sean comunes entre los individuos dentro de cada grupo de la población. Para niveles muy altos (resp., bajos) de intereses asortativos, algunas formas particulares de redes únicamente heterofílicas (resp., homofílicas) son simultáneamente estables y eficientes. Para niveles intermedios de intereses asortativos, identificamos una familia de redes de amistad, con niveles intermedios de homofilia, que son simultáneamente estables y eficientes.

Palabras claves: Redes de Amistad, Intereses Asortativos, Homofilia, Heterofilia, Diversidad, Integración.

JEL Números de Clasificación: A12, A14, D01, D71, D85, J15, Z13.

Making Friends: the Role of Assortative Interests and Capacity Constraints*

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July, 2021

Abstract

We investigate the properties of stability and efficiency of friendship networks when individuals (i) have arbitrary levels of assortative interests and (ii) are capacity-constrained in their efforts to form connections. For intermediate levels of assortative interests, extreme forms of homophilic (or heterophilic) patterns may coexist as stable with more moderate homophilic patterns. In general, though, there is a natural positive relation between levels of assortative interests and homophily levels of stable patterns. For extreme forms of homophilic (resp., heterophilic) patterns to be stable, connections between agents with different (resp., same) characteristics must have a certain minimal (good) quality as well. The homophily features of stable networks are affected by the tightness of the capacity constraint and by the discrepancies between sizes of different-characteristic groups. Efficiency requires common aggregate qualities of connections (with respect to same-characteristic agents, on one side, and with respect to different-characteristic agents, on the other side) across all individuals within each different population group. Under very high (resp., low) levels of assortative interests, some particular forms of only extreme homophilic (resp., heterophilic) patterns are simultaneously stable and efficient. For intermediate levels of assortative interests, we identify a class of friendship networks that feature intermediate levels of homophily, and for which stability and efficiency are compatible.

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The unlike is joined together, and from differences results the most beautiful harmony.

(Heraclitus of Ephesus)

1 Introduction

Friendship relationships articulate the ways in which people socialize and share information in all cultures. Based on a variety of reasons, humans choose whom they make friends with and, further, how much effort they devote to "build the quality" of such ties.

The prevalent view of the empirical literature on sociology, biology, and psychology (Lazarsfeld and Merton, 1954; Felmlee et al., 1990; Mehra et al., 1998; Christakis and Fowler, 2014; McPherson et al., 2001; Heaton, 2002) is that people tend to lean towards others with similar characteristics—a phenomenon known as *homophily*.[1] Numerous arguments have typically been given to explain the presence of high assortative interests that could underpin observed homophilic patterns: self-identity concerns, risk-sharing measures, conflict prevention, or even evolutionary selection. Nonetheless, economic motivations—based mainly on the role of complementaries—(Newman, 2001; Moody, 2004; Guimera et al., 2005; Davis et al., 2003; Watts, 1999; Uzzi, 2008) have also been proposed to rationalize more disassortative interests that could explain observed *heterophilic* friendship relations—documented mainly in the realm of scientific collaboration (Page, 2007) and of relations within organizations (Casciaro and Lobo, 2008).

Leaving aside the underlying assortative interests one might consider, all available evidence—and casual observation—supports the view that people are inherently social. Conceivably, the average person would like to devote unbounded amounts of resources to make as many friends as possible and, even further, to establish excellent-quality relations with such friends. However, we are all undoubtedly constrained in our resources (e.g., time) to make friends and to enhance the quality of our relations. *Capacity constraints* condition dramatically our decisions about friendship connections. But then, how does the underlying levels of assortative interests lead to observed homophilic patterns in the presence of capacity constraints? Can strong heterophilic connections arise in predominantly homophilic societies, or conversely? Could there exist low or intermediate levels of homophily under high levels of underlying assortative interests? What are the properties of efficient patterns when people are constrained in making friends? Can stable patterns be efficient, and under which conditions? This paper explores these questions by proposing a fairly general model where people form links that give rise to *friendship networks*.

We study a setting in which individuals are distinguished according to a certain (extrinsic) characteristic. Agents must invest quantities of an available resource in order to form friendship *links* and to determine the *qualities* of such links. The investment technology is monotone and features strategic independence between the investments made

[1] The term was coined by Lazarsfeld and Merton (1954).

by the agents. Therefore, the decision of each person is about how to allocate quantities of a fixed resource in making friends. Given a resulting friendship network, the preferences of the agents depend only on the aggregate qualities of their links. Preferences are strictly monotone and convex. We assume that a (common) level of assortative interests—which is captured by a certain (exogenous) parameter—is embedded in the agents' preferences. To fix notions, we use the term *assortative* (resp., *disassortative*) *level* to refer to the primitive that captures the degree to which agents prefer to connect with similar (resp., dissimilar) individuals. We use then the expression more or less *homophilic* (resp., *heterophilic*) to describe the extent to which connections among similar (resp., dissimilar) agents arise as stable (hence, endogenously) in friendship networks. Following the pertinent literature, high levels of assortative interests can perhaps be best justified in environments where friendship serves primarily as a socialization mean, whereas low assortative levels fit better in environments where the links are mainly used as "instrumental" means.[2] For instance, while white students in a graduate school may seek connections with other white students for pure entertainment, they may also put efforts in connections with Asian students in order to collaborate jointly in a term paper or, simply, to learn about their different culture. To encompass all such possibilities, we take a completely agnostic position relative to what are the underlying levels of assortative interests. Accordingly, this paper proposes a comprehensive exploration of stability and efficiency properties of friendship networks, for *any arbitrary level* of assortative interests.

Stability of a friendship network requires that no agent has (strict) incentives to change her aggregate investments in similar and dissimilar people (*robustness against individual deviations*) and, furthermore, that no pair of agents benefit (strictly) by changing their friendship investments (*robustness against bilateral deviations*). Efficiency of a friendship network is attained when, under the restriction of the capacity constraint imposed on the agents, the network maximizes the sum of the agents' utilities.

We model individual investments in friendship links as continuous choices within the $[0, 1]$ interval, so that efforts to build each single relationship are naturally assumed to be bounded. Given this, we are then particularly interested in identifying full-intensity investments—i.e., investments that equal one. For technical reasons, we consider that a friendship link arises even when only one of the two involved agents contributes.[3] For simplicity, we also consider that there are only two possible characteristics that the agents may have. In regard to the capacity constraint, we assume that the available resource allows each agent to invest fully in each other agent of any of the two possible characteristics, yet not in all the rest of agents in the population.

Our results naturally point towards a positive relation between (exogenous) assortative

[2] Recent empirical work on sociology suggests even that the assortative motivations of individuals can shift through their lifetime. Roughly, the view is that people seeks for "more profound" or "intimate" relationships at early age and schooling time, whereas their interests for mating shift to "more instrumental/practical" ones when work becomes central to their lives. The investigation by Thomas (2019) supports this view and, in particular, finds that racial homophily declines as people age.

[3] Our main results, though, continue to hold in specifications where mutual investments are required.

interests and (endogenous) homophilic stable patterns. Yet, even while the stability notion that we use refines most concepts commonly proposed in the literature,[4] multiple networks, with very different features in terms of homophily, may arise as stable for assortative interests that are not too extreme. Second, our main results on stability highlight a feature on the incentives of pairs to sustain friendship links, which we term as "premium of mutual efforts." In particular, under a simple monotone additive-linear technology to produce *linkage quality*, both agents in any given pair can benefit strictly if they redirect simultaneously into each other efforts devoted to other friends (outside from the pair). As a consequence, if the amount of friendship efforts that any agent could make in some other agent were unbounded, then no pattern would be stable. Intuitively, Anne and Bob could always diminish their efforts with respect to some other friends and gain by using the so saved resources in improving jointly their own relationship. But then Bob and Charles could do the same, and so on, endlessly. However, as mentioned, the amount of investments that can be made (and received) for each particular relation are bounded. As a consequence, the premium of mutual efforts ceases to have effect if one friend is already saturating what she can invest in the other. This leads to the insight that a stable friendship networks requires that, in each pair of different agents, at least one of them invests with full intensity into the other. Obviously, this condition for stability can be achieved with the proviso of the capacity constraint faced by the agents.

The role on stability played by the premium of mutual efforts tells us a lot about the friendship connections in stable patterns that feature extreme forms of homophily or heterophily. For instance, for a resulting network with extreme homophily features, which we define as *maximally homophilic*, the existing links between agents with different characteristics must be relatively intense as well. In this sense, our model delivers the message that *heterophilic connections of relatively high quality arise in extreme forms of homophilic patterns*. The analog message follows for populations that feature extreme forms of heterophilic patterns, which we define as *minimally homophilic*. In addition, when capacity constraints are relatively tight, it follows that stability of a pattern is easier to attain whenever, for each pair i, j of friends, only one of them, say i, invests with full intensity in the other agent j while, at the same time, the recipient j of such effort does not invest fully in i. In a stable pattern, such a recipient j of the full-intensity effort is incentivized to devote her own full-intensity efforts to other people $k \neq i$. In this sense, a characteristic of stable patterns is that *individuals coordinate in ways such that only one friend in each relationship acts as "main sponsor" while the other "free rides."* This message becomes starker as the capacity constraints tighten.

We show that extreme forms of homophilic and heterophilic patterns cannot coexist simultaneously under a common level of assortative interests. Also such extreme patterns are harder to sustain as stable ones when the capacity constraint tightens. Instead, tighter

[4] In particular, it refines the notions of Nash stability, Pairwise stability, and Pairwise Nash stability. Interestingly, any further refinement of the stability concept that we use would lead to that no stable network exists in our setup. In this sense, our approach to stability seeks to meaningfully reduce the multiplicity of stable networks down to a minimum.

capacity constraints make it possible for intermediate forms of homophily patterns, which we label as *partially homophilic*, to arise as stable ones. Finally our results also show that extreme forms of homophilic patterns become harder to sustain as stable for societies with high discrepancies between the sizes of different-characteristic groups. This insight is quite consistent with the implications of recent empirical analyses (e.g., Currarini et al. (2009)). Nonetheless, obtaining extreme forms of heterophilic patterns as stable ones in our setting is not restricted by any discrepancies between the sizes of different-characteristic groups.

With regard to efficiency, the homogeneity assumed in the agents preferences and capacity constraints, combined with the convexity of such preferences, lead to the key implication that an efficient friendship network must feature uniform aggregate qualities, within the agents that have a common characteristic, derived from *same-type* and *different-type* links. This implication highlights a source for inefficiency of stable networks. In summary, stability requires first that all agents comply individually with the given level of assortative interests—under the restriction of their capacity constraints. In addition to this, proposing particular stable patterns requires the construction of *minimal sets of full-intensity investments between pairs of agents* to avoid the effects of the premium of mutual efforts. Given all this, compatibility of stability and efficiency requires then that the construction of full-intensity investments be such that the induced aggregate qualities of same-type and different-type links be common across all agents of each characteristic. For extreme levels of assortative interests, complying with both requirements simultaneously, lead to the insight that only some particular forms of maximally (or minimally) homophilic stable networks are efficient. In some cases, fulfilling simultaneously both such requirements depends crucially on the sizes of different-characteristic groups.

Equipped with such a central insight, we propose then constructions of minimal sets of full-intensity investments between agents of a common characteristic that guarantee efficiency of some minimally homophilic networks, regardless of any discrepancies between the sizes of different-characteristic groups. As to maximally homophilic networks, finding the minimal set of full-intensity investments, under which stability and efficiency are compatible, becomes trickier depending on the sizes of the groups with different characteristics. We provide a construction of full-intensity investments between agents of different characteristics that ensures efficiency of some maximally homophilic networks when the two group sizes coincide. Finally, for intermediate levels of assortative interests, we also identify stable patterns that feature intermediate degrees of homophily, which we define as *partially homophilic networks*. For an intuitive class of partially homophilic networks, we also derive a construction of full-intensity investments that guarantees the efficiency of such friendship patterns.

Finally, although any (non-extreme) level of assortative interests may give rise to multiple classes of stable friendship patterns in our setup, we are able to identify particular bounds under which only either homophilic, heterophilic, or partially homophilic networks always exist as stable. Multiplicity of stable networks is a common feature in the literature on social networks. Therefore, the bounds that we provide—which ensure existence and uniqueness—can have useful implications for statistical work. Because extreme forms of homophilic behavior are extensively documented by the empirical literature, and the pertinent data can be easily obtained, an econometrician can use our results on the uniqueness of patterns with common homophily features to infer properties about underlying assortative interests in many practical scenarios.

The article is organized as follows. Section 2 outlines the baseline model. Section 3 analyzes the properties of stability of friendship networks and Section 4 focuses on their efficiency features. Section 5 comments on literature connections and Section 6 concludes. The Appendix provides proofs omitted from the main text.

2 A Model of Formation of Friendship Relationships

There is a *population* $N \equiv \{1, \ldots, n\}$ of agents that can be distinguished according to a certain (extrinsic) characteristic—e.g., ethnicity, religious affiliation, education, cultural background, profession, or age. In particular, each agent has a *type* $\theta \in \Theta \equiv \{A, B\}$ that captures the characteristic. Based on θ, the entire population N can be divided into two *groups of people* N_θ with respective sizes $n_\theta \equiv |N_\theta|$, for $\theta \in \Theta$, so that $N = N_A \cup N_B$ and $n = n_A + n_B$.[5]

We assume that $n_\theta \geq 3$ for each $\theta \in \Theta$ and, without loss of generality, set $n_A \geq n_B$ throughout. Accordingly, we will refer to group N_A as the (weakly) *larger group* and to N_B as the *smaller group*. When considering a given a type $\theta \in \Theta$, we will typically use θ' to refer to the alternative type $\theta' \neq \theta$. Also, for agent i of type θ, we will use the short-hand notation $N_\theta^i \equiv N_\theta \setminus \{i\}$ to indicate the group of agents, other than herself, that have her own characteristic.

2.1 Friendship Networks

People make decisions about forming *friendship links* that give rise to friendship networks. We view friendship links as vehicles of entertainment, communication, or collaboration in a broad sense. Two linked agents can enjoy time together, receive advice from each other, or work collaboratively. A *friendship network* g is collection $g \equiv \{g_{ij} \in [0, 1] \mid i, j \in N\}$ of *linkage qualities* $g_{ij} \in [0, 1]$ for each pair of agents $i, j \in N$. We use G to denote the set of all possible friendship networks. A linkage quality g_{ij} captures the quality of the link that goes from agent i to agent j under network g. Thus, on one extreme, $g_{ij} = 0$ indicates that i is not linked to j whereas $g_{ij} = 1$ describes a full-quality link. Furthermore, we consider *undirected networks* in which links are bidirectional so that, by construction,

[5] The model considers two possible realizations of the characteristic—and, therefore, two population groups—for simplicity. The main implications, though, continue to follow qualitatively under an arbitrary number of groups.

$g_{ij} = g_{ji}$ for each pair of agents $i, j \in N$. We consider that each agent is linked to herself with full quality, i.e., $g_{ii} = 1$.

2.2 Linking Decisions

Individuals make their linking decisions in a (simultaneous-move) network formation game. Each agent i makes simultaneously an *investment effort* $x_{ij} \in [0, 1]$ to form a friendship link with each other agent $j \neq i$ in the population. The quantity x_{ij} describes the *intensity* of i's investment to become linked with j. The assumption that $x_{ij} \in [0, 1]$ allows for "infinitesimal" investment efforts. An *investment strategy* for each agent i is a vector $x_i \equiv (x_{ij})_{j \neq i} \in [0, 1]^{n-1}$. Let $x \equiv (x_i)_{i \in N} \in [0, 1]^{n(n-1)}$ be a *strategy profile*. As usual, x_{-i} will denote a combination of strategies for all individuals other than agent i. Similarly, let x_{-i-j} denote a combination of strategies for all individuals excluding the pair of (different) individuals i and j. Thus, we can express a strategy profile x either as $x = (x_i, x_{-i})$ or as $x = (x_i, x_j, x_{-i-j})$, for $j \neq i$.

Investments in a friendship connection determine the quality of the link according to a simple additive-linear technology.

Assumption 1. Given a strategy profile x, the *linkage quality* $g_{ij}(x) = g_{ji}(x)$ of the connection between agents i and j is given by

$$g_{ij}(x) \equiv (1/2) \left[x_{ij} + x_{ji}\right]. \tag{1}$$

Investments are thus strategically independent in the technology that determines linkage quality. Note that, although links are bilateral, a single agent can form a link $g_{ij}(x) > 0$ by making a positive investment in the relationship—i.e., $g_{ij}(x) > 0$ whenever $x_{ij} > 0$ even if $x_{ji} = 0$. In other words, the formation of a friendship relation does not require a positive effort by both agents, though its quality is enhanced when both contribute. This consideration allows us to avoid discontinuities in the linkage quality technology. Let $g(x)$ denote the friendship network induced by the profile x according to the technology described by Eq. (1) above.

Given a strategy profile x and an agent $i \in N$, let $N_i(x) \equiv \{j \in N \setminus \{i\} \mid x_{ij} = 1\}$ be the set of agents, different from i, that receive full-intensity investments from agent i under the profile x. Also, for agent i of type θ, let the quantity $s_i(x) \equiv \sum_{j \in N_\theta^i} g_{ij}(x)$ capture the aggregate quality of the links that connect agent i to all other same-type agents, and, analogously, let $d_i(x) \equiv \sum_{j \in N_{\theta'}} g_{ij}(x)$ describe the total quality of the links that connect agent i to all different-type agents. When no reference need be made to the underlying strategy profile x, we will drop the x argument and simply write s_i and d_i. Let then $S_i \equiv [0, n_\theta - 1]$ and $D_i \equiv [0, n_{\theta'}]$ be the sets of possible total qualities, respectively, of same-type and different-type links for agent i of type θ. Allowing for $x_{ij} \in [0, 1]$ leads to that the variables $s_i \in S_i$ and $d_i \in D_i$ are non-negative real numbers.

2.3 Preferences

The preferences of an individual i over networks are described by a function $\pi_i : G \longrightarrow \mathbb{R}_+$. Furthermore, we consider that each agent i cares only about the total qualities (s_i, d_i) associated to her friendship links.[6] Specifically, we consider that the function π_i has the form $\pi_i(g(x)) = u(s_i(x), d_i(x))$, where $u : S_i \times D_i \longrightarrow \mathbb{R}$ captures the utility $u(s_i, d_i)$ that any agent i receives from the aggregate qualities (s_i, d_i) of her same-type and different-type friendship links. The function u is common across agents. Notice also that u does not depend on the entire architecture of the network. This assumption is relatively common in the literature on friendship networks—e.g., Currarini et al. (2009); Boucher (2015); Currarini et al. (2017), among others. Under the above considerations, Assumption 2 describes the features that we impose on preferences.

Assumption 2. For each agent $i \in N$, the utility function u is smooth and satisfies:

(1) $u(0,0) = 0$ and $u(s_i, d_i) \geq 0$ for each $(s_i, d_i) \neq (0,0)$.

(2) $u(s_i, d_i)$ is strictly increasing in (s_i, d_i).

(3) $u(s_i, d_i)$ is strictly concave in (s_i, d_i).

(4) There is a given cutoff proportion $\beta \in (0, +\infty)$ of qualities of different-type (relative to same-type) friendship links such that

   (a) $\partial u(s_i, d_i)/\partial s_i = \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i = \beta$;

   (b) $\partial u(s_i, d_i)/\partial s_i > \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i > \beta$;

   (c) $\partial u(s_i, d_i)/\partial s_i < \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i < \beta$.

Assumption 2−(1) is just for normalization. Assumption 2−(2) imposes monotonicity on the utility that each agent receives from the qualities of her friendship links. Geometrically, in the (s_i, d_i) space, the utility from any investments in friendship links increases in any ray that departs from the origin. Assumption 2−(3) imposes convexity on each agent's preferences over the (s_i, d_i) space of total friendship qualities.

Assumption 2−(4) is key to describe the way in which agents could either be (relatively) more interested in mating either same-type or different-type individuals. In short, the condition describes whether agents have either assortative or disassortative interests, as well as the degree of such interests. In particular, Assumption 2−(4) establishes that (a) there is a fixed fraction $\beta = d_i/s_i$—which geometrically corresponds to the slope of a ray going out of the origin in the space (s_i, d_i)—such that the marginal utilities from linking

[6] This is an important consideration since we are ignoring other plausible ways in which people could in principle care about the architecture of the resulting friendship network $g(x)$. In particular, we consider that agents do not care about the identity of the agents they are linked to, neither about the features of their indirect connections (i.e., features along the paths of friends of own friends).

with either type of agents are equal. Given this cutoff value β, then (b) if the proportion of qualities of different-type links (relative to same-type links) lies above the required fraction β, then the marginal utility from additional qualities of different-type links becomes lower than the marginal utility from same-type links. The converse condition is described by condition (c).[7]

Intuitively, parameter β captures the (common) level of assortative interests in the population. Values of β in the interval $(0, 1)$ correspond to situations where people lean relatively more towards assortativity, whereas values of β in the interval $(1, +\infty)$ describe situations where disassortative interests dominate.[8] We can view more assortative interests as naturally based on pure "socialization" motivations and more disassortative interests as relying on more "instrumental" motivations—usually via complementarities arising between agents of different characteristics.

2.4 Capacity Constraints

Conceivably, people would like to enjoy unbounded qualities of friendship links with others, either for pure socialization or for more instrumental reasons. We consider, though, that agents are exogenously constrained over the total intensity of their friendship investments. We assume that each agent has a total resource (e.g., of time) $R_m > 0$ (captured by a positive integer) to invest in friendship links with others. Thus, an important consideration of the model is the presence of *capacity constraints* over the total investments in friendship qualities. Specifically, we assume

Assumption 3. Each individual $i \in N$ is constrained over her friendship investments x_{ij} according to the restriction $\sum_{j \neq i} x_{ij} \leq R_m$, for a certain bound $R_m \equiv n_A + m$, where m is an integer $m \in M \equiv \{1, \ldots, n_B - 2\}$.

Notice that, since $\min_{m \in M} R_m = n_A + 1 > n_A$, Assumption 3 allows each agent to invest with full intensity in links to all other agents from either group, N_A or N_B, separately. On the other hand, since $\max_{m \in M} R_m = n_A + n_B - 2 < n - 1$, Assumption 3 also implies that no agent is able to invest with full intensity in links to all the remaining agents in the population. Of course, this is an obvious requirement to keep the model interesting under strictly monotone preferences. Higher values of m give us looser (or less tight) capacity constraints. We consider that the possible values of the total resource R_m are integer

[7] This condition can be equivalently interpreted in terms of the marginal rate of substitution of the utility function u between the aggregate qualities s_i and d_i. Geometrically, the conditions put structure on the slopes of the agent's indifference curves in the (s_i, d_i) space.

[8] Of course, while lower values of the cutoff ratio $\beta \in (0, 1)$ describe higher levels of assortative interests, higher values of $\beta \in (1, +\infty)$ describe higher levels of disassortative interests. An example of a preference specification u that satisfies all the conditions required by Assumption 2 is that given by a Cobb-Douglas function $u(s_i, d_i) = s_i^a d_i^b$ such that $a > 0$, $b > 0$, and $a + b < 1$. In this case, the level of assortative interests β described in Assumption 2–(4) is equal to $\beta = b/a$.

numbers for technical (and exposure) reasons.[9] Let $X_i \equiv \{x_i \in [0,1]^{n-1} \mid \sum_{j \neq i} x_{ij} \leq R_m\} \subset [0,1]^{n-1}$ be the set of agent i's *investment strategies under capacity constraints* and let $X \equiv \times_{i \in N} X_i \subset [0,1]^{n(n-1)}$ be the *set of all possible investment profiles under capacity constraints*.

2.5 Stability Notion

Let us use $\Gamma \equiv < N, \Theta, X, (\pi_i)_{i=1}^n) >$ to denote the network formation game that we have described. To explore stable friendship networks, we resort to a straightforward adaptation to our game of the *weak bilateral equilibrium (wBE)* stability concept proposed by Boucher (2015).

Definition 1. A *weak bilateral equilibrium (wBE)* of the network formation game Γ is a strategy profile x^* that satisfies:

1. *robustness against unilateral deviations:* for each individual $i \in N$, we have $\pi_i(g(x^*)) \geq \pi_i(g(x_i, x^*_{-i}))$ for each $x_i \in X_i$;
2. *robustness against bilateral deviations:* for each pair of (different) individuals $i, j \in N$, we have that $\pi_i(g(x_i, x_j, x^*_{-i-j})) > \pi_i(g(x^*))$ implies $\pi_j(g(x_i, x_j, x^*_{-i-j})) \leq \pi_j(g(x^*))$ for each $x_i \in X_i$ and $x_j \in X_j$.

A network g is a *stable friendship network* if there is a weak bilateral equilibrium x^* of the network formation game Γ such that $g = g(x^*)$.

Condition 1. of Definition 1 is the best-response condition required by the Nash stability notion. Condition 2. adds then the requirement that a wBE be immune, not only against (strictly) profitable unilateral deviations, but also against any possible bilateral deviation that be *strictly* beneficial to *both* agents in the pair. The notion of wBE weakens the concept of bilateral equilibrium due to Goyal and Vega-Redondo (2007). Yet, it refines most stability notions commonly used in the literature on network formation. In particular, wBE refines Nash stability (proposed by Myerson (1991)), Pairwise stability (proposed by Jackson and Wolinsky (1996)), and Pairwise Nash stability (which combines both the Nash and the Pairwise stability requirements).

As a prelude to our analysis of stability, we describe the decision problem that each agent faces when she cares only about her unilateral incentives. Since our assumptions on preferences depend only on the shape of the function u, it is useful to work with a given agent i's (unilateral) problem directly in terms of the variables s_i and d_i. For agent i of type θ, let $I_i^s(x_{-i}) \equiv (1/2) \sum_{j \in N_\theta^i} x_{ji}$ and $I_i^d(x_{-i}) \equiv (1/2) \sum_{j \in N_{\theta'}} x_{ji}$ be the (normalized)

[9] Such a discrete set of possible values $\{n_A + 1, \ldots, n_A + n_B - 2\}$ for R_m allows us to have a clear description of how investment in friendship links can be allocated in the presence of monotone preferences and capacity constraints.

total incoming intensities that agent i receives, respectively, from same-type and different-type people under the combination x_{-i}.[10] Then, for a fixed x_{-i}, the unilateral decision problem that each agent i faces is to choose x_i in a way such that the induced total qualities $s_i = s_i(x_i, x_{-i})$ and $d_i = d_i(x_i, x_{-i})$ solve the problem

$$\max_{\{s_i,d_i\}} \quad u(s_i, d_i)$$
$$\left.\begin{array}{l}\text{s.t.: } I_i^s \le s_i \le (n_\theta - 1)/2 + I_i^s; \\ \qquad I_i^d \le d_i \le n_{\theta'}/2 + I_i^d; \\ \qquad s_i + d_i \le R_m/2 + I_i^s + I_i^d \end{array}\right\} \quad D_i(x_{-i}), \tag{2}$$

where

$$D_i(x_{-i}) \equiv \big\{(s_i, d_i) \in [I_i^s(x_{-i}), (n_\theta - 1)/2 + I_i^s(x_{-i})] \times [I_i^d(x_{-i}), n_{\theta'}/2 + I_i^d(x_{-i})] \\ \text{s.t.: } \quad s_i + d_i \le R_m/2 + I_i^s(x_{-i}) + I_i^d(x_{-i})\big\}$$

gives us the set of same-type and different-type qualities (s_i, d_i) feasible for agent i, given a profile of investment strategies x_{-i} followed by the rest of agents. Let $\phi_i : X_{-i} \longrightarrow X_i$ denote the *best-response of agent* i, specified as[11]

$$\phi_i(x_{-i}) \equiv \{x_i \in X_i \mid u(s_i(x_i, x_{-i}), d_i(x_i, x_{-i})) \ge u(s_i(x_i', x_{-i}), d_i(x_i', x_{-i})) \quad \forall x_i' \in X_i\}.$$

Accordingly, let $\Phi : X \longrightarrow X$, where $\Phi = (\phi_1, \ldots, \phi_n)$, be the *best-response correspondence of all agents* in the society. Then, the Nash stability condition, imposed by 1. of Definition 1 above, can be equivalently expressed as requiring that x^* that satisfies the classical fixed point condition $x^* \in \Phi(x^*)$.

Finally, we close this section by introducing four particular values of the level β of assortative interests in the population that depend on the rest of primitives of the model and that will play important roles in the analysis of stable and efficient friendship networks. In particular, the following values of β will be quite relevant:

$$\begin{aligned} \beta_L(m) &\equiv \frac{R_m - (n_A - 1)}{2(n_A - 1)} = \frac{(m+1)}{2(n_A - 1)}, \\ \beta_l(m) &\equiv \frac{nR_m - n_A(n_A - 1) - n_B(n_B - 1)}{2n_A(n_A - 1)} = \frac{n(m+1) + n_B(n_A - n_B)}{2n_A(n_A - 1)}, \\ \beta_h(m) &\equiv \frac{n_A}{R_m - n_A} = \frac{n_A}{m}, \quad \text{and} \quad \beta_H(m) \equiv 2\beta_h(m) = \frac{2n_A}{m}. \end{aligned} \tag{3}$$

Recall that $m = R_m - n_A \in \{1, \ldots, n_B - 2\}$.

[10] We will drop the x_{-i} argument when no reference need be made to the underlying strategy combination. Note that $s_i = (1/2)\sum_{j \in N_\theta^i} x_{ij} + I_i^s$ and $d_i = (1/2)\sum_{j \in N_{\theta'}} x_{ij} + I_i^d$ for each strategy profile $x \in X$.

[11] The correspondence ϕ_i can be equivalently specified as $\phi_i(x_{-i}) \equiv \{x_i \in X_i \mid \pi_i(g(x_i, x_{-i})) \ge \pi_i(g(x_i', x_{-i})) \quad \forall x_i' \in X_i\}$.

3 Stability of Friendship Networks

We follow a two-step strategy to explore stability features of friendship networks. In the first step, we characterize (in Lemma 1) the optimal investment strategy of any given agent as a best-response to the investments strategies chosen by the rest of individuals. This step allows us to explore the unilateral incentives to deviate from a given network. By considering networks where all agents best-reply to the rest of agents, we derive Nash stable networks, as required by condition 1. of our stability notion (Definition 1). The second step consists of identifying—and thereby ruling out—plausible bilateral deviations from any given network that is already robust against unilateral deviations. The bilateral deviations ruled out in the second step (in Lemma 2) shed light on the class of deviations that can be bilaterally profitable (conditional upon networks which are already immune against unilateral deviations). Accordingly, these two steps combined provide conditions to identify stable networks according to both 1. and 2. of Definition 1

3.1 Step I–Unilateral Optimal Decisions

Let us consider a given agent i's decision problem in terms of the qualities (s_i, d_i), as described in Eq. (2). From the monotonicity condition of Assumption 2–(2), we observe that agent i wishes to choose x_i so as to induce the highest possible qualities s_i and d_i. In her decision, though, the agent is constrained by the number of available people of each type, by the incoming investments from other individuals, and by her capacity constraint as required by Assumption 3. Analyzing agent i's (unilateral) problem boils down to studying how the linear constraints that describe the feasible set $D_i(x_{-i})$ of the problem in Eq. (2) bind in plausible solutions.

The solutions to problem Eq. (2) can be easily analyzed with the aid of Fig. 1. The figure depicts in blue the feasible set $D_i(x_{-i})$ of the problem in Eq. (2). The (saturated) capacity constraint

$$s_i + d_i = R_m/2 + I_i^s + I_i^d, \tag{4}$$

which follows directly from Assumption 3, is displayed as the green dotted line in the figure. Also note that the black dotted line would correspond to a hypothetical capacity constraint for the (uninteresting) case in which we could have $R_m = n-1$—which would allow agents to be friends of everyone else with full quality. Therefore, under our consideration that $R_m < n - 1$ (Assumption 3), we know that the actual capacity constraint lies necessarily below the black dotted line and that the shape of the feasible set $D_i(x_{-i})$ is always as depicted in the figure. The rays in red correspond to three possible levels of assortativity interests $\beta > \beta' > \beta''$, as described by Assumption 2-(4). In the figure, β gives us disassortative interests, whereas β' and β'' give us two different levels of assortative interests.[12] For a fixed $x^*_{-i} \in X_i$, the optimal choice (s^*_i, d^*_i) of agent i is described by the

[12] Recall that the marginal rate of substitution of u—which geometrically gives us the slope of its

point where the highest indifference along the corresponding ray β intersects the feasible set $D_i(x^*_{-i})$.

To ease the notational burden, we find useful to set, for a given type θ of a given agent i, and for a fixed combination of strategies x_{-i}, the following values for the level β of assortativity interests:

$$\underline{\beta}(\theta; x_{-i}) \equiv \frac{R_m - (n_\theta - 1) + 2I^d_i(x_{-i})}{(n_\theta - 1) + 2I^s_i(x_{-i})} \text{ and } \overline{\beta}(\theta; x_{-i}) \equiv \frac{n_{\theta'} + 2I^d_i(x_{-i})}{R_m - n_{\theta'} + 2I^s_i(x_{-i})}. \tag{5}$$

Since u is smooth and concave, the respective solutions to the problem in Eq. (2) can always be obtained as captured by [a], [b], and [c] in the figure. The arguments above lead directly to the algebraic description, that we summarize in Lemma 1, of the unilateral optimal behavior of each individual in terms of the variables $s_i = s_i(x_i, x^*_{-i})$ and $d_i = d_i(x_i, x^*_{-i})$ as a best-response to a strategy combination x^*_{-i} chosen by the rest of individuals.

Lemma 1. Assume Assumption 1—Assumption 3, and consider a preference specification u. Take a given agent $i \in N$, and take a given strategy combination x^*_{-i} chosen by the agents other than i. Consider the unilateral problem of agent i specified in Eq. (2). Then, the solutions to such a linear problem are described by:

$$s^*_i = \frac{(n_\theta - 1)}{2} + I^s_i,\ d^*_i = \frac{R_m - (n_\theta - 1)}{2} + I^d_i \text{ if } 0 < \beta \leq \underline{\beta}(\theta; x^*_{-i}). \tag{a}$$

$$s^*_i = \left(\frac{1}{1+\beta}\right)\left(\frac{R_m}{2} + I^s_i + I^d_i\right),\ d^*_i = \beta s^*_i \text{ if } \underline{\beta}(\theta; x^*_{-i}) \leq \beta \leq \overline{\beta}(\theta; x^*_{-i}); \tag{b}$$

and

$$s^*_i = \frac{R_m - n_{\theta'}}{2} + I^s_i,\ d^*_i = \frac{n_{\theta'}}{2} + I^d_i \text{ if } \beta \geq \overline{\beta}(\theta; x^*_{-i}); \tag{c}$$

Notice that, relative to the capacity constraint in Eq. (4), solutions [a] and [c] are corner solutions, whereas [b] is an interior solution. The solutions given by [a], [b], and [c] to the problem in Eq. (2) describe the three key qualitative cases that can take agent i's (unilaterally optimal) investment—stated in terms of the total qualities (s_i, d_i), for a fixed choice x^*_{-i} by the rest of agents.

If each agent $i \in N$ chooses her investment strategy $x^*_i \in X_i$ as described in Lemma 1, for each given $x^*_{-i} \in X_{-i}$, then the resulting network $g = g((x^*_i, x^*_{-i}))$ is Nash stable. Let us denote by $NS(u) \subset G$ the set of Nash stable networks for a preference specification given by u. Existence of Nash stable networks in the proposed network formation game Γ, for any given u, follows directly from the following modeling choices: (1) $x_i \in [0, 1]$ for each agent i, (2) the presence of capacity constraints, and (3) the assumption that u is smooth and concave.[13]

indifference curve—equals one along the respective ray associated to each cutoff value β, exceeds one above

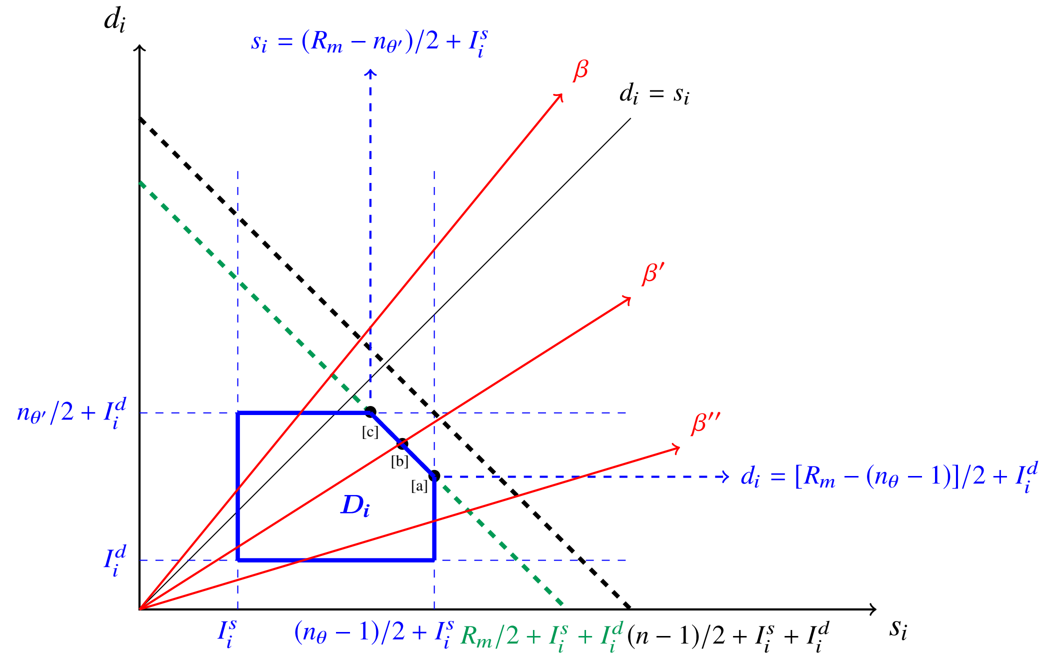

Figure 1 – Optimal choices of (s_i, d_i) for a given x^*_{-i}.

3.2 Step II–Bilateral Optimal Decisions

After ruling out unilateral deviations, stable networks follow then by preventing bilateral deviations as well, as described by condition 2. of Definition 1. Lemma 2 provides the key necessary and sufficient condition for a Nash stable network $g(x) \in NS(u)$ to be immune against bilateral deviations.

Lemma 2. Assume Assumption 1—Assumption 3, and consider a preference specification u. Consider a strategy profile $x \in X$ that induces a Nash stable friendship network $g = g(x) \in NS(u)$. Then, any given pair of two (different) agents $i, j \in N$ does not have incentives to bilaterally deviate from the profile x, as described by condition 2. of Definition 1, that is,

$$\pi_i(g(x'_i, x'_j, x_{-i-j})) > \pi_i(g(x)) \;\Rightarrow\; \pi_j(g(x'_i, x'_j, x_{-i-j})) \leq \pi_j(g(x))$$

for each $x'_i \in X_i$ and $x'_j \in X_j$ if and only if $j \notin N_i(x) \Rightarrow i \in N_j(x)$.

the ray, and it is less than one below the ray.

[13] Specifically, since u is smooth and concave, and $X \subset \mathbb{R}^{n(n-1)}$ is a compact real set, it follows that the best-response correspondence Φ of the agents in the population is upper hemi-continuous. Furthermore the correspondence Φ satisfies that $\Phi(x)$ is non-empty, closed, and convex for each profile $x \in X$. By Kakutani's fixed point theorem, we then know that, under the capacity constraints considered in Assumption 3, a Nash stable network always exists for any preference specification u that satisfies Assumption 2 in our network formation game Γ.

The logic behind the condition derived in Lemma 2 is as follows. Under the maintained assumptions, if we start from a strategy profile that induces a Nash stable network, then there could exist a unique class of bilateral deviations that be strictly beneficial to each agent from any given pair of agents in the society. This class of (potentially) profitable deviations is enabled by the consideration that $n_\theta \geq 3$ for each type $\theta \in \Theta$.

Notice first that, in any Nash stable network, each agent must exhaust the available resource R_m. Then, we consider (potential) deviations in which each of the two agents from a fixed pair were able to decrease their investments in some other agents by certain arbitrary amounts. Given this, we consider the possibility that both agents could also be able to invest the so saved amounts into each other. Because of the simple additive-linear technology for generating linkage quality considered in Eq. (1), this type of deviations would clearly be strictly profitable for both agents. At a more intuitive level, the suggested (potential) deviations capture natural situations where two friends benefit from "synergies" by mutually increasing the efforts they devote to their own relationship. We term the incentives behind this class of (potential) deviations as "premium of mutual efforts" because both agents in a given pair benefit strictly by mutually redirecting third-party investments into each other. Note that, under the capacity constraints, each of the two agents needs to lower her efforts in other friends who, given the definition of bilateral deviation, would in turn maintain their investments in them. Furthermore, under our monotonicity assumptions in the presence of capacity constraints, the only class of bilateral deviations—starting from a Nash stable network—that could result strictly beneficial to both agents in a given pair is the one described above. This is the case because, starting from a profile where all agents exhaust their resources, the only way of in which the two members of a given pair can benefit strictly is by redirecting third-party investments. Any other type of bilateral deviations would either harm or leave indifferent at least one of the agents in the pair.

To ensure then that a friendship Nash stable network is immune against this class of deviations, we need to restrict attention to strategy profiles in which at least one agent in each possible pair cannot increase any further her investment in the other agent, as stated in Lemma 2. Thus, to propose a stable friendship pattern, we need to construct a minimal set of full-intensity investments across all agents. An intuitive insight that stems form such minimal sets, in the presence of (common) capacity constraints, is then that (in stable networks x) each particular friendship relationship $g_{ij}(x)$ is mainly sponsored by only one of the two friends, say i, while the recipient j of the full-intensity investment sponsors other relationships. As the capacity constraints tighten, the recipient j of the full-intensity effort in each particular relationship reciprocates less with her friend i. Only by doing so, the recipient j is able to save amounts of the resource R_m to fulfill the required full-intensity investments in other agents $k \neq i$. The reasoning behind the above described (unique) class of (potentially) profitable bilateral deviations from Nash stable networks—which are prevented by the condition provided by Lemma 2—will be crucial to explore stable friendship networks. In particular, Proposition 1 and Proposition 2 will resort to adjusted versions of the condition stated in Lemma 2.

Importantly, notice that the class of (potentially) profitable bilateral deviations considered in the result of Lemma 2 would continue to work, given the simple linkage quality technology specified in Eq. (1), if we additionally required that $g_{ij}(x) = g_{ji}(x) > 0$ only if $x_{ij}x_{ji} > 0$—i.e., if we required that the link were formed only when both agents contribute a positive amount of investment to their relationship. Furthermore, the description that we give of the premium of mutual efforts, and the role that it plays in stability, is also robust to alternative quality technology specifications, with the restriction that such a technology be linear and with uniform slopes across the investments of all agents in the population. Under such technologies, for situations where at least one agent from a given pair does not invest fully in the other agent, both agents would benefit strictly by redirecting third-party investments because the so induced changes in aggregate qualities are linear according to a common slope. More specifically, consider a technology given by

$$g_{ij}(x) = g_{ji}(x) = A + Bx_{ij} + Bx_{ji},$$

where $A \geq 0$, $B > 0$ and $A + 2B \leq 1$. Suppose, without loss of generality, that $\beta < 1$, so that there are assortative interests in the population, and that the two agents i and j have the same type. Consider now situations where $x_{ij} < 1$ and $x_{ji} < 1$. Then, suppose that agent i decreases her investment in some other agent $k \notin \{i, j\}$ by an amount $0 < \varepsilon_i \leq 1 - x_{ij}$ and that j decreases her investment on some other agent $l \notin \{i, j\}$ by an amount $0 < \varepsilon_j < 1 - x_{ji}$. This is enabled by the assumption that $n_\theta \geq 3$ for each type $\theta \in \Theta$. The argument continues to be valid if we have $k = l$. Then, suppose that the two agents i and j invest the saved amounts ε_i and ε_j simultaneously into each other. In this case, the quality of their link increases by an amount $B(\varepsilon_i + \varepsilon_j)$. The quality of the link between agents i an k decreases by an amount $B\varepsilon_i$ and the quality of the link between agents j and l decreases by an amount $B\varepsilon_j$. For the case where k and l have also the same characteristic that agents i and j, it follows then that s_i increases by a net amount of $B\varepsilon_j$ and s_j increases by a net amount of $B\varepsilon_i$. Also, d_i and d_j remain unchanged. Since $B > 0$, this deviation is profitable to both agents i and j. To comment on other possibilities, now suppose, for instance, that k has a type different from the type of agents i. Then, if agent l has the same type that agent j, the proposed deviation implies the same change in j's utility as described above. As to agent i, the proposed deviation now entails that s_i increases by a net amount of $B(\varepsilon_i + \varepsilon_j)$ while d_i decreases by a net amount of $B\varepsilon_i$. Now, since $B > 0$ and $\beta < 1$, it follows that the proposed deviation continues to be strictly profitable for agent i. Similar arguments can be analogously provided for the cases where agents i and j have different types and/or $\beta > 1$ because each population group consists at least of three agents.

Our description of the premium of mutual efforts, however, does not go through under more general technology specifications of linkage quality technology. For instance, the identified class of (potentially) beneficial bilateral deviations does not work as considered in this paper if either $g_{ij}(x)$ were linear in x_{ij} and x_{ji} with different slopes, or if $g_{ij}(x)$

were strictly concave or convex in the agents investments. Nevertheless, our particular assumption of linear technology (with the form given by Eq. (1)) gives us a reasonable and simple formulation of how investments produce linkage quality for a continuous investment choice. Given the degree of generality that we consider on the utility function u, more complex technology specifications would obviously make intractable the exercise of deriving general properties of stable friendship patterns.

Finally, let us comment on the existence of stable networks in the proposed model. In some parts of the analysis, we will need to deal with the fact that the sizes of the population groups N_A and N_B can be either odd or even. To this end, it is useful to specify the number

$$\alpha(r) \equiv \begin{cases} r/2 \text{ if } r \text{ is even;} \\ (r-1)/2 \text{ if } r \text{ is odd} \end{cases}$$

for any given integer $r > 1$. The number $\alpha(r)$ accounts for either half of r or half of $r - 1$, depending on whether r is an even or an odd integer, respectively. A necessary condition for the investment conditions required by Lemma 2 to be satisfied is that the size of total resource R_m available to the agents be sufficiently large. In particular, if $R_m \geq \alpha(n)$, then each agent can take on the burden of full-intensity investments for (approximately) half of the population. Furthermore, $\alpha(n)$ gives the minimum size of the (common) resource that ensures that the condition required in Lemma 2 can be satisfied. Notice, though, that $R_m \geq \alpha(n)$ is always guaranteed under Assumption 3 since $\min_{m \in M} R_m = n_A + 1 > \alpha(n)$. Therefore, under Assumption 2 and Assumption 3, for each preference specification u, the necessary condition required to obtain the implication of Lemma 2 is satisfied. This, however, does not directly ensure the existence of a stable friendship network for each possible tuple (β, R_m, n_A, n_B). In our setting, the presence of (homogeneous) capacity constraints, combined with discrepancies between the groups sizes, may conflict crucially with the incentives described by the level of assortative interests β.

In particular, we make no claims regarding existence of stable patterns for "intermediate" levels of the assortative interest β. We are able, though, to guarantee existence (and uniqueness in some cases) of stable friendship networks for each possible tuple (β, R_m, n_A, n_B), under the restriction to relatively high or low (dis)assortative interests—given by β. As argued earlier (in Subsection 3.1), existence of Nash stable patterns is in fact ensured in our setting for each possible tuple (β, R_m, n_A, n_B). Yet, once the agents comply with their individual incentives (according to the level of assortative interest β), stability additionally requires the construction of minimal sets of full-intensity investments, in order to prevent profitable bilateral deviations. To illustrate the difficulties that may arise regarding existence for intermediate levels of assortative interests, consider a population in which the sizes of the two groups differ greatly and suppose that the capacity constraints that face the agents are very tight. Suppose that the level of assortative interests is intermediate and, accordingly, consider a resulting Nash stable network $g(x) \in NS(u)$ such that some agents have unilateral incentives to invest primarily in same-type fellows (as in [a] of Fig. 1) while other agents have unilateral incentives to invest amounts in both same-type and different-type agents (as in [b] of Fig. 1). In the case where the capacity constraints are as tight as possible, each agent will be able to invest with full intensity in only $\alpha(n)$ other agents. We can intuitively observe then that some agents many not be able to simultaneously comply with their individual assortative interests and meet their shares of full-intensity contributions, which are required to prevent bilateral deviations. For instance, if agents of the smaller group want to behave unilaterally as in as in [b] of Fig. 1, then they wish to invest large amounts in relations with members of a much larger group. Therefore, the investments that they devote to their heterophilic connections might leave then with not enough slack resource so as to comply with the overall minimal full-intensity requirements. As a consequence, it might well be the case a network $g(x) \in NS(u)$ be not immune against beneficial bilateral deviations.

Let us use $S(u) \subset NS(u)$ to denote the set of stable friendship networks for a given preference specification u. We will be more specific as to when we can guarantee existence of stable patterns in Subsection 3.5.

3.3 Classifying Friendship Patterns in Terms of Homophily

Our analysis of stable networks will focus on two classes of patterns where the resulting homophily levels are, respectively, very high and very low. On the one hand, we will study networks in which all agents invest with full intensity in links to all others of their same type. Given this, each agent will devote the remaining of her resource R_m to different-type agents. We will refer to such networks as *maximally homophilic* networks. On the other hand, we will consider networks in which all agents invest with full intensity in different-type agents. Then, the agents will devote the remaining of their resources to links to agents of their same type. We refer to this latter class of patterns as *minimally homophilic* networks. Note that, under the above described homophily criteria, both classes of networks yet encompass a wide variety of possible particular architectures.

The sets $N_i(x)$ introduced earlier (in Subsection 2.2) are quite useful to capture homophily features of the network $g = g(x)$. Under monotone preferences and capacity constraints, we can naturally interpret that, under $g = g(x)$, agent i is more interested in being friends with the agents in the set $N_i(x)$, relative to mating other agents that do not belong to $N_i(x)$.

Definition 2. Consider a strategy profile x that induces a friendship network $g = g(x)$. Then,

(a) the network g is said to be *maximally homophilic* if for each agent $i \in N$ of type θ, and for each type $\theta \in \Theta$, we have $N_\theta^i \subseteq N_i(x)$, and

(b) the network g is said to be *minimally homophilic* if for each agent $i \in N$ of type θ, for each type $\theta \in \Theta$ and for the alternative type $\theta' \neq \theta$, we have $N_{\theta'} \subseteq N_i(x)$.

To complete our analysis of stable patterns, we will also explore networks that feature intermediate levels of homophily—relative to maximally and minimally homophilic networks. Here, we will take the simple approach to regard a friendship network as partially homophilic whenever it is neither maximally nor minimally homophilic.

Definition 3. Consider a strategy profile x that induces a friendship network $g = g(x)$. The network g is said to be *partially homophilic* if for some type $\theta \in \Theta$, we have that either

(a) there is some agent i of type θ such that $N_i(x) \subset N^i_\theta$, with $N_i(x) \neq N^i_\theta$, or

(b) there is some agent j of type θ such that $N_j(x) \subset N_{\theta'}$, with $N_j(x) \neq N_{\theta'}$,

or both (a) and (b).

The empirical literature on social networks makes use of interesting measures to assess homophily levels using the available data. We introduce now (adjusted versions) of such measures and compare them with our notions of maximally and minimally homophilic networks. The measures given by Definition 4 below build closely upon the measures used by Currarini et al. (2009).[14] Consider a strategy profile $x \in X$ that induces a network $g = g(x)$. Let $\bar{s}_\theta(x) \equiv \sum_{i \in N_\theta} s_i(x)/n_\theta$ and $\bar{d}_\theta(x) \equiv \sum_{i \in N_\theta} d_i(x)/n_\theta$ be the average qualities of the links of type θ agents towards, respectively, same-type and different-type agents.

Definition 4. Given the network $g = g(x)$, the *homophily index* $H_\theta(x)$ of type $\theta \in \Theta$ is given by $H_\theta(x) \equiv \bar{s}_\theta(x)/[\bar{s}_\theta(x) + \bar{d}_\theta(x)]$. Then, the network $g = g(x)$ satisfies *inbreeding homophily* relative to type θ if $H_\theta(x) > n_\theta/n$. Similarly, the network $g = g(x)$ satisfies *inbreeding heterophily* relative to type θ if $H_\theta(x) < n_\theta/n$.

The term "inbreeding homophily" has been widely used by the sociological literature (Coleman, 1958; Marsden, 1987; McPherson et al., 2001) and, in particular, the criteria specified in Definition 4 have been applied to assess empirically homophily levels in populations.[15] Now, we wish to study whether our proposal of maximally (resp., minimally) homophilic networks is in consonance with inbreeding homophily (resp., heterophily). The following lemma sheds light on the relation between the two extreme homophily features proposed in this paper and the measure of inbreeding homophily/heterophily.

Lemma 3. Under the proposed model, consider that Assumption 2 and Assumption 3 hold. Consider a given available resource $R_m = n_A + m$, where $m \in \{1, 2, \ldots, n_B - 2\}$. Then:

(i) a maximally homophilic network $g = g(x)$ satisfies inbreeding homophily with respect to type θ if and only if the sizes N_A and N_B satisfy the following condition relative to the available resource: $m < (n_B - 3) + 2n_\theta/(n_A + n_B)$;

(ii) each minimally homophilic network $g = g(x)$ satisfies inbreeding heterophily with respect to each type $\theta \in \Theta$.

[14] Rather than the average number of links, though, we rely on the average quality of the links.

[15] The logic behind such criteria lies in capturing how groups tend to "inbred" in same-type friendship connections relative to their respective fractions of the entire population.

Since we are considering that $n_A \geq n_B$, it follows directly that $(n_B-3)+2n_A/(n_A+n_B) \geq n_B-2$. Given that $m \in \{1, 2, \ldots, n_B-2\}$, we observe then that any maximally homophilic network satisfies inbreeding homophily for type $\theta = A$. On the other hand, $n_A \geq n_B$ implies that $(n_B - 3) + 2n_B/(n_A + n_B) \leq n_B - 2$. Therefore, whether a maximally homophilic network satisfies inbreeding homophily for the smaller population group is unclear and it depends crucially on the size of the available resource. For instance, if the sizes of the population groups are different, and the capacity constraint is as loose as possible so that $m = n_B - 2$, then each maximally homophilic network satisfies inbreeding heterophily for the smaller group. In general, Lemma 3 (i) shows that tighter capacity constraints (i.e., lower values of R_m) facilitates that a maximally homophilic network satisfy inbreeding homophily as well for the smaller population group.[16]

3.4 Stable Friendship Networks

Armed with the previous insights about unilateral and bilateral optimal choices, we are ready now to explore stable friendship networks.

The conditions provided by Proposition 1 below characterize strategy profiles that induce stable maximally homophilic networks. Given a strategy profile $x \in X$, the following upper bound

$$\hat{\beta}(x) \equiv \inf_{i \in N_\theta,\, \theta \in \Theta} \frac{R_m - (n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ji}}{2(n_\theta - 1)}$$

on the level of assortative interests of the population will be useful to understand how assortative interests lead to maximally homophilic networks.[17]

Proposition 1. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x a strategy profile that induces a maximally homophilic friendship network $g = g(x)$. Then, the network g is stable, i.e., $g \in S(u)$, if and only if:

1. *Robustness against unilateral deviations:* the level of assortative interests of the population described by β is sufficiently high, with the particular form given by $\beta \leq \hat{\beta}(x)$.

2. *Robustness against bilateral deviations:* provided that the resource R_m is sufficiently large, with the particular form given by $R_m \geq (n-1) - n_A n_B/n$, then for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$.

Condition 1. of Proposition 1 ensures that no agent has unilateral incentives to deviate from investing with full intensity in each other same-type individual. The condition makes

[16] In particular, simple algebra shows that the condition $m < (n_B - 3) + 2n_B/(n_A + n_B)$ can equivalently be rewritten as $m < (n_B - 1) - 2n_A/(n_A + n_B)$. Since we are considering that $n_A \geq n_B$, we know that $(n_B - 1) - 2n_A/(n_A + n_B) \leq n_B - 2$. Furthermore, such a condition holds with equality only if $n_A = n_B$, whereas it holds with strict inequality otherwise.

[17] Recall that the level of assortative interests increases with the cutoff slope β. Accordingly, we derive an upper bound on the level of assortative interests by considering a lower bound on the slope β.

use of the bounds $\underline{\beta}(\theta; x_{-i})$ (for each agent $i \in N_\theta$ and each type $\theta \in \Theta$) that guarantee unilateral optimal choices as the one described by [a] in Fig. 1. In addition, contingent on the size of the resource R_m being sufficiently large, condition 2. of Proposition 1 guarantees that no pair of agents have bilateral incentives to (jointly) deviate from profiles that induce maximally homophilic networks.

Interestingly, provided that the resource R_m is relatively large, even in the presence of high assortative interests, patterns of extreme homophily features can be sustained as stable ones only if links of a certain (relatively high) quality between agents of different types arise as well. In particular, there must exist a link between each pair (i, j) of different-type agents with quality g_{ij} no less than $1/2$. In short, a certain degree of qualities of heterophilic relations is necessary to sustain maximally homophilic stable networks.

Proposition 1 characterizes investments profiles that lead to maximally homophilic networks, in terms of the primitives n_A, n_B, β, R_m. Naturally, maximally homophilic networks require that the level of assortative interests of the agents be sufficiently high. Nonetheless, multiple networks, not all of them necessarily being maximally homophilic, arise as stable ones for the levels of assortative interests captured by Proposition 1. As an antidote to such multiplicity issues, Corollary 1 gives us a bound on the level of assortative interests that guarantees the existence of only maximally homophilic networks.

Corollary 1. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x be a strategy profile that induces a friendship network $g = g(x)$. Provided that $R_m \geq (n-1) - n_A n_B/n$, suppose that x satisfies that for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$. Then, if the level of assortative interests in the population is sufficiently high, with the particular form given by $\beta < \beta_L(m)$, where $\beta_L(m)$ is the bound on β specified in Eq. (3), then the unique class of stable networks consists entirely of maximally homophilic netwoks.

The sufficient condition stated in Corollary 1 follows directly by considering a bound $\beta_L(m)$ on the cutoff values $\hat{\beta}(x)$ such that, if $\beta < \beta_L(m)$, then: (a) no partially homophilic network, as described in Definition 3, can be sustained as stable and (b) each agent would strictly prefer to invest with full intensity in all other same-type agents, as captured by [a] of Fig. 1. Upon considering an agent $i \in N_\theta$ who does not invest with full intensity in each other same-type agent, such a bound $\beta_L(m)$ can be derived by noting that $\overline{\beta}(\theta; x_{-i})$ cannot be (strictly) lower than $[R_m - (n_\theta - 1)]/2(n_\theta - 1)$, which would correspond to a (hypothetical) situation where $\sum_{j \in N_{\theta'}} x_{ji} = 0$ and $\sum_{j \in N_\theta^i} x_{ji} = (n_\theta - 1)$ (that is, agent i would receive no investments whatsoever from different-type agents and full investments from all other same-type agents). Then, if $\beta < [R_m - (n_\theta - 1)]/2(n_\theta - 1)$, it follows that even such an agent, who would be in the best position to behave unilaterally either as [c] or [b] of Fig. 1, would rather choose to invest in all other same-type agents instead, as captured by [a] in Fig. 1. Notice that the magnitude of the bound $\beta_L(m)$ considered in Corollary 1 depends crucially on the size of the available resource R_m.[18]

[18] In particular, $\beta_L(m)$ increases with R_m in the interval $\beta_L(m) \in \left[1/(n_A - 1), (n_B - 1)/2(n_A - 1)\right]$.

We would like to use the insights from Proposition 1 to explore further certain features of maximally homophilic stable networks. Note that the conditions described by the proposition yet allow for a broad class of networks to be stable. Motivated by the consideration that all agents in the society face a common available resource R_m, we pay special attention to stable networks in which the burden of full-intensity investments between the agents that belong to different groups is distributed across the agents in a relatively uniform way. Corollary 2 gives us conditions under which a class of maximally homophilic networks, with certain symmetries in the amounts invested by the agents, arise as stable ones. To this end, it is convenient to introduce first the details of a certain partition of any of the two population groups N_θ, for $\theta \in \Theta$.

Observation 1. Upon relabelling the names of the agents (say, switching indexes from i to i_k and j_k), let us partition each of two groups N_θ, for $\theta \in \Theta$, into two sets, N_θ^L and N_θ^H, according to: (a) N_A is partitioned into $N_A^L = \{i_1, \ldots, i_{\alpha(n_A)}\}$ and $N_A^H = \{i_{\alpha(n_A)}+1, \ldots, i_{n_A}\}$, whereas (b) N_B is partitioned into $N_B^L = \{j_1, \ldots, j_{\alpha(n_B)}\}$ and $N_B^H = \{j_{\alpha(n_B)} + 1, \ldots, j_{n_B}\}$.

The partition specified in Observation 1 separates each group N_θ into two subgroups, N_θ^L and N_θ^H, of the same size if the number of agents n_θ in the group is even. If the number of agents in the group N_θ is odd, then the set N_θ^H contains just one more agent than the set N_θ^L. Thus, under the obvious restriction that agents are indivisible, the sizes of the resulting subgroups N_θ^L and N_θ^H are the most similar possible ones. Provided that the level of assortative interests and the size of the resource are sufficiently large, Corollary 2 describes a class of maximally homophilic networks in which each agent from each group N_θ takes on the burden of full-intensity investments across different-type agents for (roughly) half of the agents from $N_{\theta'}$, for $\theta' \neq \theta$.[19] In particular, we construct a *minimal set of full-intensity investments* across different-type agents which are distributed across the agents in a relatively uniform way.

Corollary 2. Assume Assumption 1—Assumption 3, and consider a preference specification u. Then, provided that the capacity constraint is sufficiently loose, with the particular form $R_m \geq n_A + (n_B - 1)/2$, if the level of assortative interests is sufficiently high, with the particular form $\beta \leq [R_m + (n_B - n_A)]/2(n_A - 1) = (m + n_B)/2(n_A - 1)$, then there exists a class of strategy profiles $x \in X$, invariant to any relabelling of the names of the agents, which induces maximally homophilic stable networks $g = g(x) \in S(u)$.

In particular, given the partitions of groups detailed in Observation 1, such a class of strategy profiles x can be described as: (1) each agent $i \in N_A^L$ invests with full intensity $x_{ij} = 1$ in each agent $j \in N_B^L$, and each agent $i \in N_A^H$ invests with full intensity $x_{ij} = 1$ in each agent $j \in N_B^H$, whereas (2) each agent $j \in N_B^L$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^H$, and each agent $j \in N_B^H$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^L$.[20]

[19] Of course, if the number of agents in each group N_A and N_B is even, then each agent agent takes on the burden of full-intensity investments for exactly half of the agents from the different-type group.

[20] Given the partitions of groups of people into subgroups according to Observation 1, conditions (1)

We will use several examples (Example 1–Example 3) to illustrate our main results on stability.

Example 1. —*A Maximally Homophilic Network.*

Consider a population $N = \{1, \ldots, 7\}$ such that $N_A = \{1, 2, 3, 4\}$ and $N_B = \{5, 6, 7\}$. Notice then that $\alpha(n_A) = \alpha(4) = 2$ and $\alpha(n_B) = \alpha(3) = 1$. Therefore, Assumption 3 imposes a capacity constraint to the agents such that $R_m \in \{5\}$—i.e., we are restricted to considering just one possible value R_1=5 for the total resource. Hence, each agent will be able to invest with full intensity in five other agents in the society. Regarding the bound on the level of assortative interests identified by Corollary 1, note that $\beta_L(1) = 1/3$. In addition, we have that $2n_B/n + (n_B - 3) = 6/7 < 1 = m$. Thus, it follows from the implications of Lemma 3 that a maximally homophilic network will satisfy inbreeding heterophily for the smaller group N_B.

Following Observation 1, consider that the group N_A is divided into two subgroups, $N_A^L = \{1, 2\}$ and $N_A^H = \{3, 4\}$. Similarly, the group N_B is separated into two subgroups, $N_B^L = \{5\}$ and $N_B^H = \{6, 7\}$. Let us consider a maximally homophilic network where each agent from group N_A makes full-intensity investments in each of the other three agents in her same subgroup, whereas each agent from group N_B makes full-intensity investments in each of the other two agents in her same subgroup. In addition, using the description provided by Corollary 2, consider that each agent from the subgroup $N_A^L = \{1, 2\}$ makes a full-intensity investment in the (unique) agent from the group $N_B^L = \{5\}$, whereas each agent from the subgroup $N_A^H = \{3, 4\}$ makes full-intensity investments in each agent from the subgroup $N_B^H = \{6, 7\}$. On the other hand, consider that agent 5 (the unique member of the subgroup N_B^L) makes full-intensity investments in each agent from the subgroup $N_A^H = \{3, 4\}$, while each agent from the subgroup $N_B^H = \{6, 7\}$ makes full-intensity investments into each agent from the subgroup $N_A^L = \{1, 2\}$. Given the description of the profile thus far, it can be easily verified that, for each of the twelve possible pairs $(i, j) \in N_\theta \times N_{\theta'}$ of different-type agents, we have that one agent, either $i \in N_\theta$ or $j \in N_{\theta'}$, invests with full intensity in the other agent. Thus, as required by Lemma 2, for each of the $7 \times 6 = 42$ possible pairs of agents in the society, at least one of the two agents makes a full-intensity investment in the other agent. Therefore, no pair of agents have incentives to deviate, complying with condition 2. of Definition 1.

Given the description provided thus far, notice that while agents 3 and 4 are exhausting their 5 units of resource, the rest of agents in the society are only investing 4 units of the resource. Therefore, agents in the subgroups N_A^L, N_B^L, and N_B^H still wish to allocate their remaining 1 unit in further friendship connections. To complete the description of a strategy profile for this example, let us consider then that (i) each agent $i \in N_A^L = \{1, 2\}$ invests $1/2$ units in each agent $j \in N_B^H = \{6, 7\}$, (ii) agent 5 invests $1/2$ units in each

and (2) of Corollary 2 above lead to that each agent $i \in N_\theta$ must invest with full intensity in links to either $(n_\theta - 1) + \alpha(n_{\theta'})$ or $(n_\theta - 1) + [n_{\theta'} - \alpha(n_{\theta'})]$ other agents in the population, depending on whether $i \in N_\theta^L$ or $i \in N_\theta^H$. This captures a relatively uniform distribution of efforts by the agents to contribute the formation of links between connecting the two different subgroups.

agent $i \in N_A^L = \{1,2\}$, and (iii) each agent $j \in N_B^H = \{6,7\}$ invests $1/2$ units in each agent $i \in N_A^H = \{3,4\}$. This gives us a possible way to consider a uniform distribution of investments across different-type agents. As a result, each link between each pair of agents $(i,j) \in N_A \times N_B$ features quality $g_{ij} = 3/4$. For this (now completely described) strategy profile x, we can derive the ratio $d_i(x)/s_i(x)$ for each agent $i \in N$ as follows:

$$d_i/s_i = (1+5/4)/3 = 3/4 \text{ for } i \in N_A^L; \; d_i/s_i = (1+1)/3 = 2/3 \text{ for } i \in N_A^H;$$
$$d_j/s_j = (3/2+1)/2 = 5/4 \text{ for } j \in N_B^L; \; d_j/s_j = (3/2+3/2)/2 = 3/2 \text{ for } j \in N_B^H.$$

Therefore, for values of the level of assortative interests $\beta \in (0, 2/3]$ all agents behave individually as described by the solution [a] in Fig. 1. We can thus guarantee that the proposed strategy profile, which induces a maximally homophilic network, is immune both against unilateral and bilateral deviations. Indeed, for the particulars of this example, notice the condition on assortative interests stated in Corollary 2 requires that

$$\beta \leq \frac{R_m + (n_B - n_A)}{2(n_A - 1)} = \frac{5 + (3-4)}{2(4-1)} = \frac{2}{3}.$$

Now, with regard to extreme forms of heterophilic patterns, Proposition 2 provides conditions that characterize when minimally homophilic networks arise as stable ones. Given a strategy profile $x \in X$, the following lower bound

$$\tilde{\beta}(x) \equiv \sup_{i \in N_\theta,\, \theta \in \Theta} \frac{2n_{\theta'}}{(R_m - n_{\theta'}) + \sum_{j \in N_\theta^i} x_{ji}}$$

on the level of assortative interests of the population will be useful to grasp how assortative interests lead to maximally homophilic networks.[21]

Proposition 2. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x a strategy profile that induces a minimally homophilic friendship network $g = g(x)$. Then, the network g is stable, i.e., $g \in S(u)$, if and only if:

1. *Robustness against unilateral deviations:* the level of assortative interests of the population described by β is sufficiently low, with the particular form given by $\beta \geq \tilde{\beta}(x)$.

2. *Robustness against bilateral deviations:* provided that the resource R_m is sufficiently large, with the particular form given by $R_m \geq n_A + (n_B - 1)/2$, then for each pair of agents from a common group, $i, j \in N_\theta$, with $i \neq j$, for each type $\theta \in \Theta$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$.

Condition 1. of Proposition 2 ensures that no agent has unilateral incentives to deviate from investing with full intensity in links to each different-type individual. The condition

[21] Recall that, in order to derive a lower bound on the level of assortative interests, we need to an upper bound on the slope β.

makes use of the bounds $\overline{\beta}(\theta; x_{-i})$ (for each agent $i \in N_\theta$ and each type $\theta \in \Theta$) that guarantee a unilateral optimal choice as the one described by [c] in Fig. 1.

In addition, contingent on the size of the resource R_m being sufficiently large, condition 2. of Proposition 2 guarantees that no pair of agents have bilateral incentives to (jointly) deviate from profiles that induce minimally homophilic networks.

Note that the conditions described by the proposition yet allow for a quite broad class of networks to be stable. A converse insight to the one provided by Proposition 1 follows from Proposition 2. Provided that the resource R_m is relatively large, even in the presence of low assortative interests, extreme forms of heterophilic patterns can be sustained as stable ones only if same-type agents i, j build connections among them whose linkage qualities g_{ij} be no less than $1/2$. A certain degree of quality of the homophilic relations is necessary to sustain minimally homophilic stable networks.

In a way totally analogous to our investigation of maximally homophilic networks, Corollary 3 provides a bound on the level of assortative interests that guarantees the existence of only minimally homophilic networks.

Corollary 3. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x be a strategy profile that induces a friendship network $g = g(x)$. Provided that $R_m \geq n_A + (n_B - 1)/2$, suppose that x satisfies that for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$. Then, if the level of assortative interests in the population is sufficiently low, with the particular form given by $\beta > \beta_H(m)$, where $\beta_H(m)$[22] is the bound on β specified in Eq. (3), then the unique class of stable networks consists entirely of minimally homophilic netwoks.

The sufficient condition stated in Corollary 3 follows directly by considering a bound $\beta_H(m)$ on the cutoff values $\tilde{\beta}(x)$ such that, if $\beta > \beta_H(m)$, then: (a) no partially homophilic network can be sustained as stable, and (b) each agent would strictly prefer to invest with full intensity in all different-type agents, as captured by [c] of Fig. 1. Upon considering an agent $i \in N_\theta$ who does not invest with full intensity in each different-type agent, such a bound $\beta_H(m)$ can be derived by noting that $\overline{\beta}(\theta; x_{-i})$ cannot be (strictly) higher than $2n_{\theta'}/[R_m - n_{\theta'}]$, which would correspond to a (hypothetical) situation where $\sum_{j \in N_{\theta'}} x_{ji} = n_{\theta'}$ and $\sum_{j \in N_\theta^i} x_{ji} = 0$ (that is, agent i would receive no investments whatsoever from the same-type agents and full investments from all different-type agents). Then, if $\beta > 2n_{\theta'}/[R_m - n_{\theta'}]$, it follows that even such an agent, who would be in the best position to behave unilaterally either as [a] or [b] of Fig. 1, would rather choose to invest in all other same-type agents instead, as captured by [c] of Fig. 1.

Corollary 4 provides conditions that ensure the existence of stable minimally homophilic networks where the distribution of full-intensity investments across agents is

[22] The magnitude of the bound $\beta_H(m)$ considered in Corollary 3 decreases with the size R_m of the available resource in the interval $\beta_H(m) \in \left[2n_A/(n_B - 2), 2n_A\right]$.

relatively uniform, conditional on their characteristics. It will be useful to set the (type-dependent) integer $l_\theta \equiv \max\{n_\theta - n_{\theta'}, 0\}$.

Corollary 4. Assume Assumption 1—Assumption 3, and consider a preference specification u. Then, provided that the total resource R_m available to the agents is sufficiently large, under the particular requirement that $m \in \{\alpha(n_B), \ldots, n_B - 2\}$, if the level of assortative interests β satisfies $\beta \geq \beta_l(m)$, where $\beta_l(m)$ is the bound on β specified in Eq. (3), then there exists a class of strategy profiles $x \in X$, invariant to any relabelling of the names of the agents, which induces minimally homophilic stable networks $g = g(x)$.

In particular, such a class of strategy profiles x can be constructed as follows: for each type $\theta \in \Theta$, (1) upon relabelling the names of the agents in N_θ, set $N_\theta \equiv \{i_1, i_2, \ldots, i_{n_\theta}\}$, (2) for each agent $i_k \in N_\theta$, let then $N_{i_1}(x) = N_{\theta'} \cup \{i_2, \ldots, i_{1+(m+l_\theta)}\}$, $N_{i_2}(x) = N_{\theta'} \cup \{i_3, \ldots, i_{2+(m+l_\theta)}\}$, and so on iteratively, until reaching $N_{i_{n_\theta}}(x) = N_{\theta'} \cup \{i_1, \ldots, i_{m+l_\theta}\}$.[23]

Notice that any profile from the class described in Corollary 4 satisfies the key condition (required by Lemma 2) that $j \notin N_i(x)$ must imply $i \in N_j(x)$, for each pair of different agents $i, j \in N$—in order to prevent profitable bilateral deviations. Also, the proposed family of profiles entails that each agent $i \in N_A$ invests with full intensity in each of the n_B different-type agents and in $m + (n_A - n_B)$ same-type agents—since, in this case, we have $l_A = \max\{n_A - n_B, 0\} = n_A - n_B$. On the other hand, each agent $i \in N_B$ invests with full intensity in each of the n_A different-type agents and in m same-type agents—since $l_B = \max\{n_B - n_A, 0\} = 0$. For the proposed class of profiles, since the available resource R_m is common to the agents, it follows that the agents who belong to the largest group can spare more of their resource to fully invest in same-type agents after investing (with full intensity) in all different-type agents. Notice that we are proposing a construction for the required minimal set of full-intensity investments where the burden of investments across same-type agents is distributed uniformly.

Example 2. —*A Minimally Homophilic Network.*

Exactly as in Example 1, consider a population $N = \{1, \ldots, 7\}$ such that $N_A = \{1, 2, 3, 4\}$ and $N_B = \{5, 6, 7\}$. Recall that $\alpha(n_A) = \alpha(4) = 2$ and $\alpha(n_B) = \alpha(3) = 1$, and that Assumption 3 imposes a capacity constraint to the agents such that $R_1 = 5$. As in Example 1, each agent will be able to invest with full intensity in five other agents in the society. Recall also that the bound on the level of assortative interests identified by Corollary 1 equals $\beta_L(1) = 1/3$. Regarding the bound on the level of assortative interests identified by Corollary 3, note that $\beta_H(1) = 8$.

[23] To appreciate better the set of agents who receive full investments by each agent of each subgroup in the corollary, it is useful to consider that the agents from each set $N_\theta \equiv \{i_1, i_2, \ldots, i_{n_\theta}\}$ are arranged in a circular fashion. Then each agent i_k invests with full intensity in each of the following $i_{k+1}, \ldots, i_{k+(m+l_\theta)}$ agents. We continue in this way until each of the last agents from list $\{i_1, i_2, \ldots, i_{n_\theta}\}$ invests fully in the subsequent agents and also in the first agents from the list—thus, after completing the circumference—until completing full investments in $m + l_\theta$ agents.

Note first that $l_A = 1$ and $l_B = 0$. Then, using the construction proposed by Corollary 4, we can consider a strategy profile x such that: $N_1(x) = N_B \cup \{2, 3\}$, $N_2(x) = N_B \cup \{3, 4\}$, $N_3(x) = N_B \cup \{4, 1\}$, $N_4(x) = N_B \cup \{1, 2\}$, $N_5(x) = N_A \cup \{6\}$, $N_6(x) = N_A \cup \{7\}$, and $N_7(x) = N_A \cup \{5\}$. Given this description of the profile, it can be easily verified that, for each of the twelve possible pairs $(i, j) \in N_A \times N_A$, $i \neq j$, exactly one agent invests with full intensity in the other agent. Similarly, for each of the six possible pairs $(i, j) \in N_B \times N_B$, $i \neq j$, exactly one agent invests with full intensity in the other agent. As required by Lemma 2, for each of the $7 \times 6 = 42$ possible pairs of agents in the society, at least one of the two agents makes a full-intensity investment in the other agent. Therefore, no pair of agents have incentives to deviate, complying with condition 2. of Definition 1. Under the description provided, note that all agents in the society are using exactly their 5 units of resource. In this way, we consider a uniform distribution of investments across same-type agents. As a result each link between each pair of different agents $(i, j) \in N_\theta \times N_\theta$ from a common subgroup N_θ features quality $g_{ij} = 1/2$. Now, we can derive the ratio $d_i(x)/s_i(x)$ for each agent $i \in N$ as follows: for each agent $i \in N_A$, we have $d_i/s_i = 3/2$, whereas for each agent $j \in N_B$, we have $d_j/s_j = 4/1$. For values of the level of assortative interests $\beta \geq 4$, we can guarantee that each agent behaves individually as described by solution [c] in Fig. 1. Thus, the proposed strategy profile, which induces a minimally homophilic network, is immune both against unilateral and bilateral deviations. For the particulars of this example, the condition on assortative interests stated in Corollary 4 requires that $\beta \geq \beta_l(m) = n_A/m = 4/1 = 4$, for $n_A = 4$ and $m = 1$.

The results provided by Proposition 3 are quite useful to complement our picture of stable friendship networks. If dissasortative interests are predominant (perhaps due to that agents value more plausible complementaries arising from diversity), then maximally homophilic networks are not stable. Conversely, societies where assortative interests dominate do not feature stable minimally homophilic networks.

Proposition 3. Assume Assumption 1—Assumption 3, and consider a preference specification u. Then,

(i) if the interests for making friends β of the society lean towards disassortativity, with the particular form $\beta \in (1, +\infty)$, then there is no stable maximally homophilic network;

(ii) if the interests for making friends β of the society lean towards assortativity, with the particular form $\beta \in (0, 1]$, then there is no stable minimally homophilic network.

Intuitively, under the presence of capacity constraints, (i) if the level of assortative interests is low—so that agents value marginal investments in different-type agents more than in same-type individuals—, then agents choose not to devote the scarce resource to invest with full intensity in all other same-type agents. When group sizes are asymmetric, members of the larger group will be relatively more constrained in this respect because they are required to invest in a relatively higher number of agents under the description of a maximally homophilic network. Similarly, (ii) if the level of assortative interests is high, then agents prefer not to devote the scarce resource to invest with full intensity in all other different-type agents. Members of the smaller group will in this case be relatively more constrained in this respect.

Finally, the insights of Corollary 5 allow us to give a full description of stable friendship networks. In particular, with a flavor similar to that of the results in Corollary 1 and Corollary 3, Corollary 5 provides an interval for the level of assortative interests that guarantees the existence of only partially homophilic networks.

Corollary 5. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x be a strategy profile that induces a friendship network $g = g(x)$. Suppose that x satisfies that for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$. Then, if the level of assortative interests in the population β is intermediate, with the particular form given by $\beta_l(m) < \beta < \beta_h(m)$, then the unique class of stable networks consists entirely of partially homophilic netwoks.

3.5 Main Takeaways on Stable Patterns

At this point, we can combine the results of Proposition 1—Proposition 3 and those of Corollary 1—Corollary 5 to establish the key bounds that describe how homophily levels in stable networks depend on the level of assortative interests in the society. Our analysis has shown that:

1. for values $\beta \in (0, \beta_L(m))$, *only* maximally homophilic networks *are* stable;
2. for values $\beta \in [\beta_L(m), \beta_l(m)]$, both maximally homophilic and partially homophilic networks *may be* stable;
3. for values $\beta \in (\beta_l(m), \beta_h(m))$, *only* partially homophilic networks *may be* stable;
4. for values $\beta \in [\beta_h(m), \beta_H(m)]$, both minimally homophilic *are* stable and partially homophilic networks *may be* stable;
5. for values $\beta \in (\beta_H(m), +\infty)$, *only* minimally homophilic *are* stable.

We can now be specific about when existence and uniqueness of classes of stable networks can be guaranteed in our setting. Corollary 2 showed that if the level of assortative interests is relatively high, with the particular form $\beta \leq [R_m+(n_B-n_A)]/2(n_A-1)$, then there exists a strategy profile x that induces a class of stable maximally homophilic networks $g = g(x)$. Since $\beta_L(m)$ can be written as $[R_m + (1 - n_A)]/2(n_A - 1)$, it can be directly noted that $\beta_L(m) < [R_m + (n_B - n_A)]/2(n_A - 1)$ in our setting. Therefore, existence of stable patterns are guaranteed for assortative interests $\beta \leq \beta_L(m)$. In addition, Corollary 4 granted that if the level of assortative interests is relatively low, with the particular form $\beta \geq \beta_h(m)$, then there exists a strategy profile x that induces a class of stable minimally homophilic networks $g = g(x)$. The implications on existence and uniqueness follow them by combining the implications of Corollary 1 and Corollary 2 (on the side of extreme homophilic patters), and of and Corollary 3 and Corollary 4 (on the side of extreme heterophilic patterns).

Using the details of Example 1 and Example 2, recall that the bounds on the level of assortative interests captured by Corollary 1, Corollary 3, and Corollary 5 take the respective values $\beta_L(1) = 1/3$, $\beta_H(1) = 8$, $\beta_l(1) = 17/24$, and $\beta_h(1) = 4$. Therefore, for those examples we know that only maximally homophilic networks are stable for values $\beta \in (0, 1/3)$, whereas only minimally homophilic networks are stable for values $\beta \in (8, +\infty)$. Only partially homophilic networks are stable for $\beta \in (17/24, 4)$. Also, both maximally homophilic and partially networks coexist as stable ones for values $\beta \in [1/3, 17/24]$, whereas both minimally homophilic and partially homophilic networks coexist as stable ones for $\beta \in [4, 8]$. In particular, recall that we have described a maximally homophilic network for assortative levels $\beta \in (0, 2/3]$ (Example 1), and a minimally homophilic network for $\beta \in [4, +\infty)$ (Example 2).

Our results convey the natural message that resulting homophily levels in friendship networks are positively related with the assortative interests of the individuals in the society. However, our insights on (potentially) beneficial bilateral deviations (Lemma 2)—via the premium of mutual efforts—also show that links between each pair of agents with different characteristics must also be sponsored by (at least) one of the two friends in order to sustain extreme forms of homophilic patterns. This message that homophily does not arise in a way fully isolated (from heterophilic links of a certain quality) is also consistent with our results that, for assortative levels $\beta \in [\beta_L(m), \beta_l(m)]$, maximally homophilic networks coexist with partially homophilic ones. Such messages also extend to our investigation of extreme forms of stable heterophilic patterns. We observe that strong forms of heterophilic patterns cannot arise unless certain quality levels of homophilic connections are also present. In consonance with that message, note also that, for assortative levels $\beta \in [\beta_h(m), \beta_H(m)]$, both minimally and partially homophilic networks coexist. Finally, note that our model delivers the insight that extreme forms of homophilic and heterophilic patterns cannot coexist simultaneously under a common level of assortative interests.

Since $\beta_L(m)$ increases in m (i.e., in R_m) and $\beta_H(m)$ decreases in m (i.e., in R_m), it follows that the set of assortative levels $[\beta_L(m), \beta_H(m)]$, under which partially homophilic networks are stable, increases—according to the set inclusion order—as the capacity constraints tighten—i.e., as the resource R_m decreases. Thus, the message is that stable patterns where only any of the two extreme forms of homophily connections are present (either maximally or minimally homophilic networks) become less likely when the agents are more constrained in their resource to form links. In addition, we observe that $\beta_l(m)$ increases in m (i.e., in R_m) and $\beta_h(m)$ decreases in m (i.e., in R_m). Therefore, the set $(\beta_l(m), \beta_h(m))$ also increases—according to the set inclusion order—as the capacity constraints tighten. In short, tighter capacity constraints expands the set of assortative levels under which *only* partially homophilic patterns are stable.

As to the role played by the relative sizes of the two population groups, N_A and N_B, notice first that if $n_A = n_B$, then $\beta_l(m) = 2\beta_L(m)$ and $\beta_l(m)$ increases as the difference $n_A - n_B$ of the two population sizes rises. Therefore, it is harder for populations with high discrepancies between their group sizes to sustain maximally homophilic networks as stable ones, compared to societies that feature similar sizes for their groups. Notably, this insight is quite consistent with some results of the empirical analysis conducted by Currarini et al. (2009). Using measures the inbreeding homophily and data from Add Health (1994) for an American high school, their analysis highlights that relatively large groups, such as white and black students, tend to form friendship relations with others of their same characteristic. On the other hand, when much smaller groups are present, such as Hispanic students, they link more with students of different characteristics. These observations do not fit our notion of maximally homophilic networks because it requires that members of each different group link extensively among them, creating independent communities. In their data, it is precisely the presence of large discrepancies between group sizes what causes connections to fail to give extreme homophily patterns in the entire student population. This empirically obtained message goes in the same direction as our results on stability of maximally homophilic patterns when groups are very different in their sizes. Finally, from the expression of $\beta_H(m) = 2\beta_h(m)$, we directly observe that the bounds that allow for extreme forms of heterophilic patterns to stable ones are not restricted in any way by plausible discrepancies between the sizes of the groups with different characteristics.

We turn now to explore a particular class of partially homophilic networks that can be stable for "intermediate" assortative levels $\beta \in [\beta_L(m), \beta_H(m)]$.

3.6 A Class of Partially Homophilic Networks

An interesting special case of partially homophilic networks is that in which *all* agents behave unilaterally as described by the interior solution [b] in Lemma 1 (i.e., the solution [b] depicted in Fig. 1). In such networks no agent invests with full intensity neither in all their same-type fellows nor in all the different-type agents. In general, though, it turns out difficult to guarantee the existence of such partially homophilic networks as stable ones. Certain symmetry properties in the primitives of the model are needed. For the particular case where both population groups have a common even size, we provide a method, in Observation 2 below, to construct a family of strategy profiles that induce stable partially homophilic networks with the above mentioned feature. In addition, our proposal seeks to distribute as uniformly as possible the burden of full-intensity investments across the agents in the population.

Observation 2. We restrict attention to those populations such that $n_A = n_B = n/2$ for $n/2$ even. Upon relabelling the names of the agents in the population, let us set $N_A \equiv \{i_1, i_2, \ldots i_{n_A}\}$ and $N_B \equiv \{j_1, j_2, \ldots j_{n_B}\}$. Consider that the agents from each of the two lists $\{i_1, i_2, \ldots i_{n_A}\}$ and $\{j_1, j_2, \ldots j_{n_B}\}$ are arranged in a circular fashion. In addition, exactly as proposed in Observation 1, let us consider a partition of each of the two population groups N_θ, for $\theta \in \Theta$, into two sets, N_θ^L and N_θ^H, according to: (a) N_A is partitioned into $N_A^L = \{i_1, \ldots, i_{\alpha(n_A)}\}$ and $N_A^H = \{i_{\alpha(n_A)} + 1, \ldots, i_{n_A}\}$, whereas (b) N_B is partitioned into $N_B^L = \{j_1, \ldots, j_{\alpha(n_B)}\}$ and $N_B^H = \{j_{\alpha(n_B)} + 1, \ldots, j_{n_B}\}$.

Given those ingredients, the suggested method consists of two steps. In the first step, we describe the minimal set of full-intensity investments which guarantees that no pair of agents have bilateral incentives to deviate (as required by Lemma 2). Given this, the second step describes how agents invest in the remaining agents. In some cases, particular investment profiles could entail that some agents invest with full intensity in other agents, beyond the requirements of the first step. The underlying logic, however, is that the investments captured by the second step seek to adjust the required remaining investments so that each agent ultimately exhaust her available resource while, at the same time, the induced profile is such that each agent behaves unilaterally as the solution [b] in Lemma 1. In general, the investments that follow from our second step will be lower than full-intensity investments.

First Step.— In regard to same-type fellows, consider that, for each type $\theta \in \Theta$, each agent $i \in N_\theta$ invests with full intensity in the subsequent $\alpha(n_\theta)$ agents from the same-type list following the suggested circular arrangement. As to how agents invest with full intensity in different-type agents, consider that (a) each agent $i \in N_A^L$ invests $x_{ij} = 1$ in each agent $j \in N_B^L$; (b) each agent $i \in N_A^H$ invests $x_{ij} = 1$ in each agent $j \in N_B^H$; (c) each agent $j \in N_B^L$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^H$, and (d) each agent $j \in N_B^H$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^L$.

Notice that, in this first step, we are not giving any details about non full-intensity investments to propose our class of strategies. Observe, though, that the above description already guarantees the condition required by Lemma 2 to prevent profitable bilateral deviations from the profile x. In particular, all agents invest with full intensity in $\alpha(n_A) + \alpha(n_B) = \alpha(n)$ other agents. Then, the described full-intensity investments in the subsequent $\alpha(n_\theta)$ same-type agents (along the circular arrangement) ensure that, for each pair of same-type agents, at least one of them is investing with full intensity in the other agent. In addition, the crossed-investments among different type-agents suggested simply replicate the description proposed in Observation 1 to guarantee the robustness against bilateral deviations of the class of profiles described in Corollary 2. Corollary 2 showed that such cross-investments involving the four population subgroups ensured that, for each pair of different-type agents, at least one of them invests with fully intensity into the other.

Although we still need to describe a way to propose the pending investments so that each agent exhausts her resource, note that our description thus far entails that each agent behaves unilaterally as described by the interior solution [b] in Fig. 1. Let $\hat{N}_i(x)$ be the minimal set of full-investments of agent i constructed as suggested above. In general, we have $\hat{N}_i(x) \subseteq N_i(x)$, though it could be the case that such an inclusion relationship ultimately holds strictly in some particular cases.

Second Step.—Let us reconsider the condition over total qualities $d_i/s_i = \beta$, which is required for agent i to makes an optimal (unilateral) investment choice as the one described by [b] in Fig. 1. Given our description of the first step, such a condition can be rewritten as

$$\frac{|\hat{N}_i(x) \cap N_{\theta'}| + |\{j \in N_{\theta'} \mid i \in \hat{N}_j(x)\}| + \sum_{j \in N_{\theta'} \setminus \hat{N}_i(x)} x_{ij} + \sum_{\{j \in N_{\theta'} \mid i \notin \hat{N}_j(x)\}} x_{ji}}{|\hat{N}_i(x) \cap N_\theta^i| + |\{j \in N_\theta^i \mid i \in \hat{N}_j(x)\}| + \sum_{j \in N_\theta^i \setminus \hat{N}_i(x)} x_{ij} + \sum_{\{j \in N_\theta^i \mid i \notin \hat{N}_j(x)\}} x_{ji}} = \beta. \quad (6)$$

In addition to the requirements in Eq. (6), we must also ensure that, for each agent $i \in N$, we have

$$\sum_{j \notin \hat{N}_i(x)} x_{ij} = R_m - |\hat{N}_i(x)| \quad (7)$$

so that all agents are able to exhaust the available resource.

Notice that the method suggested in Observation 2 does not provide a closed algorithm of general applicability. However, it gives us a plausible strategy to tackle the problem of proposing particular partially homophilic networks. All the above ingredients of our method in Observation 2 to construct partially homophilic networks are illustrated in Example 3.

Example 3. —*A Partially Homophilic Network.* Consider a population consisting of eight agents such that half of them have one characteristic or the other. Let us simply index such a population as $N = \{1, 2, 3, 4\} \cup \{5, 6, 7, 8\}$, with $N_A = \{1, 2, 3, 4\}$ and $N_B = \{5, 6, 7, 8\}$. In this case, we can have two possible values for the available resource $R_m \in \{5, 6\}$, i.e., $m \in \{1, 2\}$. Let us then consider a total resource $R_1 = 5$ for this example. Also, notice we have $\alpha(n) = \alpha(8) = 8/2 = 4$, and $\alpha(n_A) = \alpha(n_B) = \alpha(4) = 4/2 = 2$.

By resorting to the partitions of each population group described in the first step of Observation 2, consider that the group N_A is divided into two subgroups, $N_A^L = \{1, 2\}$ and $N_A^H = \{3, 4\}$. Similarly, the group N_B is separated into two subgroups, $N_B^L = \{5, 6\}$ and $N_B^H = \{7, 8\}$. Then, the class of investment profiles described in Observation 2 requires that we set $\hat{N}_1(x) = \{2, 3\} \cup \{5, 6\}$, $\hat{N}_2(x) = \{3, 4\} \cup \{5, 6\}$, $\hat{N}_3(x) = \{4, 1\} \cup \{7, 8\}$, $\hat{N}_4(x) = \{1, 2\} \cup \{7, 8\}$, $\hat{N}_5(x) = \{6, 7\} \cup \{3, 4\}$, $\hat{N}_6(x) = \{7, 8\} \cup \{3, 4\}$, $\hat{N}_7(x) = \{8, 5\} \cup \{1, 2\}$, and $\hat{N}_8(x) = \{5, 6\} \cup \{1, 2\}$. Notice that, up to here, no agent is investing with full intensity neither in all of the remaining same-type agents, nor in all of the different-type agents, as required in the class of partially homophilic networks that we are considering. Now, using the condition in Eq. (6) of the second step of Observation 2 for a level of assortative interest β, we need to consider

$$
\begin{aligned}
\beta &= \frac{4 + x_{17} + x_{18} + x_{51} + x_{61}}{4 + x_{14} + x_{21}} = \frac{4 + x_{27} + x_{28} + x_{52} + x_{62}}{4 + x_{21} + x_{32}} \\
&= \frac{4 + x_{35} + x_{36} + x_{73} + x_{83}}{4 + x_{32} + x_{43}} = \frac{4 + x_{45} + x_{46} + x_{74} + x_{84}}{4 + x_{43} + x_{14}} \\
&= \frac{4 + x_{51} + x_{52} + x_{35} + x_{45}}{4 + x_{58} + x_{65}} = \frac{4 + x_{61} + x_{62} + x_{36} + x_{46}}{4 + x_{65} + x_{76}} \\
&= \frac{4 + x_{73} + x_{74} + x_{17} + x_{27}}{4 + x_{76} + x_{87}} = \frac{4 + x_{83} + x_{84} + x_{18} + x_{28}}{4 + x_{87} + x_{58}}.
\end{aligned}
$$

In addition, Eq. (7) requires that we add the constraints

$$
\begin{aligned}
&x_{14} + x_{17} + x_{18} = 1, x_{21} + x_{27} + x_{28} = 1, x_{32} + x_{35} + x_{36} = 1, x_{43} + x_{45} + x_{46} = 1, \\
&x_{58} + x_{51} + x_{52} = 1, x_{65} + x_{61} + x_{62} = 1, x_{76} + x_{73} + x_{74} = 1, x_{87} + x_{83} + x_{84} = 1.
\end{aligned}
$$

Notice that the bounds for the level of assortative interests identified in Corollary 1 and Corollary 3 take the values $\beta_L(1) = 1/3$ and $\beta_H(1) = 8$ in this example. We must then propose values of the level of assortative interests $\beta \in [\beta_L(1), \beta_H(1)] = [1/3, 8]$ to ensure that the so derived network $g = g(x)$ is robust against unilateral deviations. Given this, for the particular value $\beta = 8/7$, we can then propose symmetric non full-intensity investments so that for each agent $i \in N$, we have $x_{ij} = 1/3$ for each $j \notin \hat{N}_i(x)$.

4 Efficiency of Friendship Networks

Our analysis of efficiency properties relies on a classical *utilitarian* approach where the social planner gives all agents the same importance, regardless of their identities and characteristics.[24] In particular, we assume that the *(social) value of friendship networks* is described by a function $v : G \longrightarrow \mathbb{R}_+$, specified as

$$
v(g(x)) \equiv \sum_{i \in N} \pi_i(g(x)). \tag{8}
$$

The notion of efficiency that we use follows closely Jackson and Wolinsky (1996) who consider that a social network is efficient if it yields the highest possible social value. In addition, we naturally require the social planner to face the same capacity constraints that restrict the agents' choices. Formally,

[24] Utilitarian approaches have been commonly pursued in literature that explores the relationship between stable and efficient networks. See, among others, Jackson and Wolinsky (1996), Calvó-Armengol (2003), Goyal and Vega-Redondo (2007), Bloch and Jackson (2007), and, Bloch and Dutta (2009).

Definition 5. A friendship network $\hat{g} = g(\hat{x})$ induced by an investment profile $\hat{x}$ is efficient if, conditional on considering investment profiles that satisfy the capacity constraints, the investment profile $\hat{x}$ maximizes the sum of the utilities of all the agents in the population, that is, if $v(\hat{g}(\hat{x})) \geq v(g(x))$ for $\hat{x} \in X$ and for each $x \in X$.

In regard to the value function considered in Eq. (8), recall that we are considering that individual preferences $\pi_i(g(x))$ have the form $\pi_i(g(x)) = u(s_i(x), d_i(x))$, for a fairly general utility function $u(s_i, d_i)$ that satisfies Assumption 2. Given that our assumptions on preferences depend on the shape of the utility function u, we find it convenient to work with the efficiency notion in terms of the variables s_i and d_i. In addition, since the social planner seeks to maximize the sum of the agents' utilities, we can restrict attention to the class of friendship networks the all agents exhaust the available resource R_m.

Proposition 4 shows that any efficient pattern must necessarily have common resulting qualities of both same-type and different-type friendship links across all individuals within each of the two population groups. The key insight provided by Proposition 4 exploits the assumptions that preferences are common across agents and that they are (strictly) convex in the (s_i, d_i) space (Assumption 2–(3)). Using such assumptions, the logic of the result in Proposition 4 relies on the implication that for each feasible investment profile $x \in X$, we can find another feasible profile $\hat{x} \in X$—which can be related to x in a precise way—such that: (i) the same-type $s_i(\hat{x})$ and different-type $d_i(\hat{x})$ qualities are constant across all agents within each population group N_A and N_B, and (ii) the social value derived from $\hat{x}$ is no less than the one derived from x. Importantly, it also follows that $v(g(\hat{x})) > v(g(x))$ unless the profile x features also common qualities $s_i(x)$ and $d_i(x)$ across all agents within each population group.

Proposition 4. Assume Assumption 2 and Assumption 3, and consider a preference specification u. Let $\hat{x}$ be an investment profile that induces an efficient network $\hat{g} = g(\hat{x})$. Then, the total qualities $(s_i(\hat{x}), d_i(\hat{x}))$ must be common across all agents in each of the two population groups, that is, $s_i(\hat{x}) = s_\theta(\hat{x})$ and $d_i(\hat{x}) = d_\theta(\hat{x})$ for each agent $i \in N_\theta$ and each type $\theta \in \Theta$.

The above implication simplifies greatly our problem of finding investment profiles associated to efficient networks. Proposition 4 enables us to restrict attention to a particular family of investment profiles $\hat{x}$ that are the only candidates to induce an efficient network. Heuristically, as can be noted from the proof of Proposition 4, a profile $\hat{x}$ that can possibly induce an efficient network is characterized by the following proposal of aggregate investments. For each agent $i \in N_\theta$ and each type $\theta \in \Theta$, let

(a) $\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = y_{\theta\theta}$, and

(b) $\sum_{j \in N_{\theta'}} x_{ij} = y_{\theta\theta'}$ and $\sum_{j \in N_{\theta'}} x_{ji} = z_{\theta\theta'}$.

The family of suggested profiles $\hat{x}$ specified above is the unique family able to induce qualities $(s_i(\hat{x}), d_i(\hat{x}))$ that be constant across all agents within each population group. It follows that $s_i(\hat{x}) = y_{\theta\theta}$ and $d_i(\hat{x}) = (1/2)[y_{\theta\theta'} + z_{\theta\theta'}]$ for each $i \in N_\theta$ and each $\theta \in \Theta$. For such a family of profiles, we must consider the capacity constraints imposed on the agents (Assumption 3) with equality, so that $y_{\theta\theta} + y_{\theta\theta'} = R_m$ for each type $\theta \in \Theta$ and for the type $\theta' \neq \theta$. Finally, as also indicated in the proof of Proposition 4, such aggregate investments must also satisfy $n_A z_{AB} = n_B y_{BA}$ and, similarly, $n_B z_{BA} = n_A y_{AB}$.[25] By putting together all the considerations above, we are left with a tractable description of the problem that faces the social planner. In particular, such a problem is that of choosing profiles $\hat{x}$ in order to maximize the expression

$$\begin{aligned} v(g(\hat{x})) =& n_A u\big(y_{AA}, (1/2n_A)(nR_m - n_A y_{AA} - n_B y_{BB})\big) \\ &+ n_B u\big(y_{BB}, (1/2n_B)(nR_m - n_A y_{AA} - n_B y_{BB})\big) \end{aligned} \tag{9}$$

for the value function.

In short, we can analyze the investment profiles $\hat{x}$ that satisfy the necessary condition provided by Proposition 4—required to induce efficient networks—in terms of the aggregate outgoing/incoming investments $y_{\theta\theta} = \sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji}$ within each population group. In turn, such aggregate investments yield also the associated aggregate outgoing/incoming investments between the two different population groups:

$y_{\theta\theta'} = \sum_{j \in N_{\theta'}} x_{ij} = R_m - y_{\theta\theta}$ and

$z_{\theta\theta'} = \sum_{j \in N_{\theta'}} x_{ji} = (n_{\theta'}/n_\theta)[R_m - y_{\theta'\theta'}]$

Given all the ingredients above, we derive sufficient conditions, in Proposition 5, that characterize unique classes of investment profiles that induce efficient networks. Each class is described by the above mentioned aggregate outgoing/incoming investments $y_{\theta\theta}$, for each $\theta \in \Theta$. Nevertheless, note that each class includes multiple profiles $\hat{x}$ because the derived conditions do not depend on the particular investments $\hat{x}_{ij}$ from each agent i to another agent j in her same population group. The expression derived in Eq. (9) is key to derive the sufficient conditions provided by Proposition 5.

Proposition 5. Assume Assumption 2 and Assumption 3, and consider a preference specification u. Let $\hat{x}$ be an investment strategy profile that satisfies the necessary condition given by Proposition 4. Then,

(i) if the level of assortative interests in the population is sufficiently high, with the particular form given by $\beta < \beta_l(m)$, then the investment profile $\hat{x}$ that induces a unique

[25] This is obtained by equalizing the aggregate investments that all agents from a group N_θ make in all agents from the other group $N_{\theta'}$ to the aggregate investments that the agents from $N_{\theta'}$ receive from all agents from N_θ

class of efficient networks $g = g(\hat{x})$ satisfies

$$\text{for } i \in N_\theta, \quad \sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = n_\theta - 1, \quad \sum_{j \in N_{\theta'}} x_{ij} = R_m - (n_\theta - 1), \text{ and}$$

$$\text{for } i \in N_A, \quad \sum_{j \in N_B} x_{ji} = (n_B/n_A)[(m+1) + (n_A - n_B)];$$

$$\text{for } i \in N_B, \quad \sum_{j \in N_A} x_{ji} = (n_A/n_B)(m+1).$$

(ii) if the level of assortative interests in the population is sufficiently low, with the particular form given by $\beta > \beta_h(m)$, then the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies, for each $i \in N_\theta$ and each $\theta \in \Theta$, $\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = R_m - n_{\theta'}$, $\sum_{j \in N_{\theta'}} x_{ij} = n_{\theta'}$, and $\sum_{j \in N_{\theta'}} x_{ji} = n_{\theta'}$.

Notice that the class of investment profiles $\hat{x}$ identified in (i) of Proposition 5 correspond to maximally homophilic networks, whereas the ones identified in (ii) of Proposition 5 correspond to minimally homophilic networks. For all agents $i \in N_\theta$, the profiles identified in (i) give us common qualities

$$s_i(\hat{x}) = n_\theta - 1 \text{ and } d_i(\hat{x}) = \frac{n(m+1) + n_B(n_A - n_B)}{2n_\theta}.$$

Similarly, the profiles identified in (ii) deliver common qualities $s_i(\hat{x}) = R_m - n_{\theta'}$ and $d_i(\hat{x}) = n_{\theta'}$ for all $i \in N_\theta$. We observe from our definitions of extreme forms of homophily patterns (Definition 2), that only maximally homophilic networks are efficient for assortative levels $\beta < \beta_l(m)$, whereas only minimally homophilic networks are efficient for values $\beta > \beta_h(m)$.

A few insights emerge from Proposition 4 and Proposition 5. We observe first a certain discrepancy between stability and efficiency of partially homophilic networks. In particular, although partially homophilic networks are stable for assortative interests $\beta \in [\beta_L(m), \beta_l(m)] \cup [\beta_h(m), \beta_H(m)]$, such networks are not efficient. Secondly, let us go back to our main examples and explore the efficiency of the friendship networks there constructed. Notice that the maximally homophilic network constructed in Example 1 does not satisfy the necessary condition that the resulting aggregate qualities (s_i, d_i) be common across all agents within each population group. In that example, we indeed derived

$$d_i = 9/4 \text{ for } i \in N_A^L \text{ whereas } d_i = 2 \text{ for } i \in N_A^H, \text{ and}$$
$$d_j = 5/2 \text{ for } j \in N_B^L \text{ whereas } d_j = 3 \text{ for } j \in N_B^H.$$

Thus, such a network is not efficient. Note that the description given in (i) Proposition 5 of the unique class of maximally homophilic efficient networks requires that each agent j from the population group N_B receives an aggregate investment $\sum_{j \in N_A} x_{ji} = (n_A/n_B)(m+1)$ from the agents from the population N_A. The model's assumptions imply directly that $(n_A/n_B)(m+1) \geq 2$, with strict inequality if $n_A > n_B$. Therefore, we obtain the property that if the population sizes are not equal, then for a maximally homophilic network to be efficient, at least three agents from the larger group N_A must invest a positive amount into each agent from the smaller group N_B. This property is not satisfied in Example 1 since agent $5 \in N_B$ is receiving investments from only two agents from the group N_A—i.e., those agents in $N_A^L = \{1, 2\}$.

The inefficiency of the maximally homophilic network of Example 1 highlights a more general feature of maximally homophilic networks when the population groups differ in their sizes. Conditional on the agents of each group having invested with full intensity in all their same-type fellows, then stability requires that at least one agent within each pair of different-type agents invest with full intensity in the other agent. Attaining such minimum full-intensity investments naturally entails asymmetries in investments between the two different groups when their sizes are different. Then, when the two groups differ in their sizes, the efficiency requirement that the aggregate qualities d_i be common across all agents from each group become harder to achieve. When the two groups have the same size, however, there are no asymmetries between the investments between different groups required to attain stability of a maximally homophilic network. In fact, Observation 3 gives a method to construct maximally homophilic networks that are simultaneously stable and efficient, provided that the groups of agents with different characteristics have the same size.

Observation 3. Consider a situation where $n_A = n_B$. Suppose that the level of assortative interests is sufficiently high, with the particular form $\beta \leq (m+1)/(n_A - 1) = 2\beta_L(m)$.

Upon relabelling the names of the agents in the two population groups, let us set $N_A \equiv \{i_1, i_2, \ldots, i_{n_A}\}$ and $N_B \equiv \{j_1, j_2, \ldots, j_{n_B}\}$. Consider a class of strategy profiles x described as follows. For each agent $i_k \in N_A$, let

$$N_{i_1}(x) = N_A^{i_1} \cup \{j_2, \ldots, j_{1+(m+1)}\},$$

$$N_{i_2}(x) = N_A^{i_2} \cup \{j_3, \ldots, j_{2+(m+1)}\},$$

and so on iteratively, until reaching

$$N_{i_{n_A}}(x) = N_A^{i_{n_A}} \cup \{j_1, \ldots, j_{1+m}\}.$$

Analogously, for each agent $j_k \in N_B$, let

$$N_{j_1}(x) = N_B^{j_1} \cup \{i_1, \ldots, i_{1+m}\},$$

$$N_{j_2}(x) = N_B^{j_2} \cup \{i_2, \ldots, i_{1+(m+1)}\},$$

and so on iteratively, until reaching

$$N_{j_{n_B}}(x) = N_B^{j_{n_B}} \cup \{i_{n_A}, \ldots, i_{n_A+m}\}.$$

Under this proposal, each agent of type $\theta = A$ invests with full intensity in exactly $R_m - (n_A - 1) = m + 1$ agents of type $\theta = B$. In addition, each agent of type $\theta = A$ receives also $R_m - (n_A - 1) = m + 1$ units from the agents of type $\theta = B$. Notice that the available resource $R_m = n_A + m$, with $m \in \{\alpha(n_A), ..., n_B - 2\}$, is sufficiently large to allow for each pair of different-type agents to have at least one of them investing with full intensity into the other agent. In fact, the class of described strategy profiles x satisfies the key condition in Lemma 2 and, therefore, any induced network $g = g(x)$ is robust to bilateral deviations. Under the considered condition on the level of assortative interests $\beta \leq (m+1)/(n_A-1)$, the networks in the induced class are also robust against unilateral deviations. Furthermore, we obtain the resulting qualities $s_i(x) = n_\theta - 1$ and $d_i(x) = R_m - (n_\theta - 1)$ for each agent of type $i \in N_\theta$, for each type $\theta \in \Theta$. Any network constructed in this way satisfy the necessary condition for efficiency required by Proposition 4 that the qualities (s_i, d_i) be constant across all agents within each population group. Finally, notice that, for the particular case $n_A = n_B$, we have that $(m + 1)/(n_A - 1) = 2\beta_L(m) = \beta_l(m)$. Therefore, the class of constructed networks satisfy also the sufficient condition in (i) of Proposition 5, which ensures efficiency.

For the case of minimally homophilic networks the stability condition that, for each pair of agents, at least one of them invests fully in the other needs to be satisfied within each group. This requirement contrasts sharply with what is needed for the case of maximally homophilic networks. In particular, this requirement entails no asymmetries in investments within each group, even when the groups differ greatly in their sizes. In such cases, finding a minimally homophilic network that be simultaneously stable and efficient is always guaranteed, as detailed in Observation 4. In fact, the class of minimally homophilic networks constructed in Example 2 satisfies the necessary condition required by Proposition 4. Furthermore, using the details of this example, we can verify that $\beta_h(1) = 4$. Those minimally homophilic networks suggested in the example were stable for values of the assortative interests $\beta \geq 4$. Thus, the sufficient condition given by Proposition 5 guarantees that the class of minimally homophilic networks constructed in Example 2 are efficient. Furthermore, for $\beta > 4$, the investment profile used to construct the network in the example belongs to the unique class of profiles that induce efficient networks.

Observation 4. Suppose that the level of assortative interests in the population is sufficiently low, with the particular form $\beta \geq n_A/m = \beta_h(m)$. Let us resort to the class of minimally homophilic networks constructed in Corollary 4. First, upon relabelling the names of the agents in N_θ, for each type $\theta \in \Theta$ and the type $\theta' \neq \theta$, let us set $N_\theta \equiv \{i_1, i_2, \ldots, i_{n_\theta}\}$. Then, for each agent $i_k \in N_\theta$, let us consider $N_{i_1}(x) = N_{\theta'} \cup \{i_2, \ldots, i_{1+m}\}$, $N_{i_2}(x) = N_{\theta'} \cup \{i_3, \ldots, i_{2+m}\}$, and so on iteratively, until reaching $N_{i_{n_\theta}}(x) = N_{\theta'} \cup \{i_1, \ldots, i_m\}$.

Regarding stability, note that the available resource $R_m = n_A + m$, where we have $m \in \{\alpha(n_B), ..., n_B - 2\}$, is sufficiently large to allow for each pair of same-type agents to enjoy a full-investment made by (at least) one of the two agents in the pair. Any network in the suggested class is therefore robust to bilateral deviations. Moreover, while each agent

$i \in N_\theta$ is investing exactly $R_m - n_{\theta'}$ units in her same-type fellows, she is also receiving exactly $R_m - n_{\theta'}$ units of investment from the agents in her own population group. Thus, for the proposed level of assortative interest $\beta \geq \beta_h(m)$, the networks in the suggested class are also robust to unilateral deviations.

Regarding efficiency, notice that for each agent $i \in N_\theta$ and each type $\theta \in \Theta$, it follows that the quality of her different-type links is $d_i(x) = n_{\theta'}$, for $\theta' \neq \theta$, while the quality of her same-type links is $s_i = R_m - n_{\theta'}$. The networks in this class satisfy then the necessary condition for efficiency in Proposition 4 that the qualities (s_i, d_i) be common across all agents within each population group. Furthermore, for $\beta \geq \beta_h(m)$, the suggested class of networks satisfies also the sufficient condition in (ii) of Proposition 4. The proposed class of mimimally homophilic networks are thus stable and efficient.

For the particular case where the sizes of both population groups are the same, the following corollary to Proposition 5 allows us to fully characterize efficient networks in terms of the assortative interests of the population.

Corollary 6. Assume Assumption 2 and Assumption 3, and consider a preference specification u. Then,

(i) the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies, for each $i \in N_\theta$ and each $\theta \in \Theta$,

$$\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = n/2 - 1, \quad \sum_{j \in N_{\theta'}} x_{ij} = R_m - (n/2 - 1), \quad \text{and} \quad \sum_{j \in N_\theta} x_{ji} = (m + 1)$$

if and only if level of assortative interests in the population is sufficiently high, with the particular form given by $\beta < \beta_l(m)$;

(ii) the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies, for each $i \in N_\theta$ and each $\theta \in \Theta$, $\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = R_m - n/2$, and $\sum_{j \in N_{\theta'}} x_{ij} = \sum_{j \in N_{\theta'}} x_{ji} = n/2$ if and only if the level of assortative interests in the population is sufficiently low, with the particular form given by $\beta > \beta_h(m)$;

(iii) if the level of assortative interests in the population is intermediate, with the particular form given by $\beta \in (\beta_l(m), \beta_h(m))$, then the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies $\hat{y} = y_{AA} = y_{BB}$ with $\beta = (R_m - \hat{y})/\hat{y} = (n + 2m)/2\hat{y} - 1$ and, therefore, $\hat{y} = R_m/(1 + \beta) = (n + 2m)/2(1 + \beta)$ for an aggregate investment choice $\hat{y} \in (R_m - n/2, n/2 - 1)$.

Going back to our examples, recall that the stable partially homophilic network which was constructed in Example 3 (under the class explored in Subsection 3.6) required a level of assortative interests $\beta = 8/7$. Now, it can be also verified that the network obtained in the example features $\hat{y} = \sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = 7/3$ for each agent $i \in N_\theta$ and each type $\theta \in \Theta$. We observe from the implication in (iii) of Corollary 6 that efficiency requires in this case that $\hat{y} = R_m/(1+\beta) = 5(1+8/7) = 7/3$. Hence, such a partially homophilic network is both stable and efficient.

5 Literature Connections

The stability notion that we use in this paper builds closely upon the *weak bilateral equilibrium (wBE)* stability concept proposed by Boucher (2015), which, in turn, weakens the concept of bilateral equilibrium due to Goyal and Vega-Redondo (2007). As commented in fn. 4, the set of stable networks in our set up would be empty if we were to use instead the (slightly strongest) bilateral equilibrium notion. The notion that we use, though, allows us to reduce crucially the multiplicity of stable networks usually present in the theoretical literature. Our approach to analyze efficient friendship networks follows the canonical framework proposed by Jackson and Wolinsky (1996).

On the instrumental side, our proposal where agents make continuous-investment choices that determine the "strength," or quality, of the link that connects them can also be found in Bloch and Dutta (2009). Another similarity with Bloch and Dutta (2009) lies in considering a fixed amount of a resource that the agents can allocate in their link formation efforts. Their model is quite different, though, in the questions explored. In particular, they do not consider agents with different characteristics and, accordingly, neither analyze questions regarding homophily.[26] The investigation of frameworks where agents do have different characteristics has indeed been an important topic in the social and economic networks literature. Most efforts have traditionally focused on the question of how assortative interests influence the outcomes of relevant network-based phenomena, such as decisions in labor markets (Montgomery, 1991), opinion formation (Golub and Jackson, 2012), friendship formation via matching (Currarini et al., 2009), formation of random networks (Bramoullé et al., 2012), or strategic network formation (De Marti and Zenou, 2017; Iijima and Kamada, 2017).

Perhaps the closest paper to ours in terms of the type of questions asked is Currarini et al. (2009) who propose a search model of endogenous matching to explore friendship connections. As in our model, in their setting agents care ultimately only about same-type and different-type links. Their exercise is quite different from ours as their goal is to match (and rationalize) certain empirical regularities regarding homophily. We attempt to provide a theoretical framework, in which plausible assortative interests are taken as a primitive, that helps us understand properties of patterns with relevant homophilic or heterophilic features in the presence of capacity constraints. At the modeling level, we use a simultaneous-move network formation game, while their model is one of dynamic

[26] The pioneering contributions on strategic link formation within the economic and social networks literature for the case where agents are not distinguished according to (extrinsic) characteristics are Jackson and Wolinsky (1996) and Bala and Goyal (2000). Other contemporary efforts include Goyal and Vega-Redondo (2007), Hagenbach and Koessler (2010), Galeotti et al. (2013), and Baumann (2021).

matching. Also within the empirical literature, Mele (2017) proposes a model of network formation in which, as in our setting, agents are divided into two different categories. For the case of sufficiently large networks, his analysis provides useful identification and estimation techniques. Another paper related to ours is De Marti and Zenou (2017), which adapts the symmetric connections model by Jackson and Wolinsky (1996) to a setting in which individuals may have two types and linking costs are endogenous. Unlike our setup, link formation is done through a discrete choice in their model. Also, they consider the stability notion of pairwise stability, which would lead to a profound multiplicity of stable patterns if applied to our setting. Another recent paper, quite different from ours in terms of the questions asked and the setup proposed, in which agents differ ex ante in their characteristics is Galeotti et al. (2006).

Baccara and Yariv (2013) consider a model in which homophily arises endogenously as a consequence of a (binary) project choice. Similarly to our model, a given parameter determines the (exogenous) inclination of the agents for one project or another. In their model, stability requires that agents connect sufficiently with (relatively) similar individuals. The authors also provide conditions under which connections between dissimilar agents arise in stable patterns. In addition, for an application in which the projects allow for information sharing, their analysis conveys the message that segregation is easier to maintain when the preferences of the individuals between the two projects are sufficiently opposed. Although their model is quite different form ours, their perspective of studying endogenous homophily levels that may arise from quite general (exogenous) tastes resembles our approach to the topic.

Finally, some of our messages relative to the stability of heterophilic friendship patterns are reminiscent of the insights provided by Galenianos (2021). His model is quite different from ours as he does not consider general friendship connections but focuses on the formation of referral networks in job markets. As a consequence, the motivations of the agents to form links are very specific to job market situations. In particular, workers form links in order to refer to and be refereed by according to the demands of firms. Interestingly, referral networks in his setup feature high levels of heterophily, with the particular form of being hierarchical.[27]

6 Concluding Remarks

This paper has developed a framework to explore stability and efficiency properties of friendship networks in populations of agents with different characteristics. We have taken any plausible underlying level of assortative interests as a primitive of the model. Additionally, we have assumed that investments in each single relationship are bounded and that the agents are capacity-constrained in the amounts of investments they can make

[27] Recent empirical work on labor markets (Hensvik and Skans, 2016; Beaman et al., 2018) offers findings very consistent with such results of hierarchical networks of referrals.

relative to the rest of the population. The proposed setting has the flavor of traditional (static) consumption/production choice models. The decision choice that faces each agent when assessing her unilateral incentives resembles a classical utility-maximization problem, though the feasibility constraint has different fundamentals and form. The presence of capacity constraints stands out as crucial consideration. In those elements, our proposal is perhaps quite different from most available models in the literature on social networks. However, our results complement some views offered by the recent papers that deal with homophily and segregation in groups. We conclude by discussing how our insights can be regarded in consonance with key implications of a few pieces of available evidence and models.

In their approach to match empirically observed patterns of inbreeding homophily, Currarini et al. (2009), argue that a bias in preferences towards same-type agents (alongside with a bias in the matching process proposed in their setting) is needed. Without such a bias, the larger group would feature inbreeding homophily while the smaller group would necessarily exhibit inbreeding heterophily. Under our notion of extreme form of homophilic patterns (maximally homophilic networks), our Lemma 3 implies that the larger group features always inbreeding homophily. This is clearly in consonance with the results by Currarini et al. (2009). In contrast, whether the smaller group exhibits inbreeding homophily or not depends crucially on the capacity constraint. This is a natural implication in our setting because the slack of resources (after investing with full intensity in same-type agents) that individuals can invest in different-type agents conditions crucially the degree of homophily that a stable network can have. The capacity constraint is, therefore, particularly critical in determining the homophilic behavior of the smallest group. Note that the members of the smaller group devote relatively fewer resources (compared to the members of the larger group) to invest in their same-type fellows—due to the discrepancies between population sizes. Given this, if the capacity constraint is relatively loose, then the agents from the smaller group can invest a relatively high amount of slack resources in different-type agents. As a consequence, the overall pattern of connections exhibits inbreeding heterophily. Intuitively, if they have the means and time for it, members of small groups can interact relatively more with agents with different characteristics, even in overall extremely homophilic patterns. The message that agents from smaller groups tend to have relatively more connections with people of different characteristics has also been pointed out by Blau (1977).

Our Lemma 3 also shows that both population groups exhibit inbreeding heterophily in our stable minimally homophilic networks. Simultaneous inbreeding heterophily for both large and small population groups is not in accordance with the empirically observed patterns documented in Currarini et al. (2009). Such patterns of extreme forms of heterophilic behavior are also at odds with most findings of the sociological literature on homophily issues in groups (e.g., McPherson et al. (2001)). However, as argued in some parts of the paper, the levels of disassortative interests required for such minimally homophilic networks to be stable in our setting are in consonance with those observed in more "practical" connections, mainly in the professional sphere. Here we should recall that the Add Health data used by Currarini et al. (2009) corresponds to a High School, thus naturally capturing connections where assortative interests would be primarily driven by socialization, self-identification, or pure entertainment motivations. The assortative levels typically present in such real-world environments would lead to that no minimally homophilic network is stable in our setting. As to the welfare analysis, Currarini et al. (2009) invoke particular forms for the agents' utilities. Under such forms, they obtain that, provided that (i) same-type and different-type links are substitutes, and (ii) the marginal benefits of same-type links are the highest possible, a pattern of complete segregation maximizes welfare. This insight is clearly in consonance with our result that, for high enough assortative interests, maximally homophilic networks are efficient.

The approach followed by Baccara and Yariv (2013) considers a setting in which an agent's type captures her inclination towards either of two public projects. Then, in stable situations, agents that are similar in their inclinations must end up in the same endogenous group. Their general model thus delivers a certain degree of (endogenous) homophily. In addition, for an application in which connections allow for information sharing, their results point towards that fully segregated groups composed by agents of the same type can emerge only when types are sufficiently different. Although our modeling choice is very different, their implication is in consonance with our result that maximally homophilic networks arise as stable only if interests for making friends lean strongly towards assortativity—i.e., $\beta \in (0, \beta_l(m)]$. The analysis of Baccara and Yariv (2013) also obtains that stable groups may be heterogeneous with the particular form that such patterns must not contain only one type of individual. In this vein, our result that stable maximally homophilic networks are characterized by a certain degree of quality of heterophilic connections is also in consonance.

Finally, let us comment on a plausible modification to enrich the setting proposed in this paper, and the scope of results. This modification consists of introducing heterogeneity in the strength of the assortative interests of the agents. This would have strong implications for the emergence of stable structures. In particular, the resulting new setting would be incapable of deriving stable structures in which all individuals behave as in solution [b] in Fig. 1. Stable partially homophilic networks would instead require that some individuals behave as in solutions [c] or [a] in Fig. 1—due to that condition $\beta s_i = d_i$ would no longer be required for all agents for a common β. Interestingly, even under heterogeneous assortative interests, our results in Proposition 1 and Proposition 2 would continue to hold with minor modifications. In particular, consider, without loss of generality, that $\beta_1 > \beta_2 > \cdots > \beta_n$. Then, the result in Proposition 1 would continue to hold if $\beta_1 \leq \hat{\beta}(x)$—i.e., if the individual with the lowest assortative interests values relatively more same-type links than different type ones. Similarly, the result of Proposition 2 would continue to hold if $\beta_n \geq \tilde{\beta}(x)$—i.e., if the individual with the highest assortative interests values relatively more different-type links than same-type ones.

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Appendix

Omitted Proofs.—

Proof of *Lemma 2*. Consider a strategy profile x that induces a friendship network $g = g(x)$. Since $n_\theta \geq 3$ for each type $\theta \in \Theta$, note that we can consider a deviation by a pair of agents $i, j \in N$, for $i \neq j$, described as follows.

(i) Agent i decreases the sum of her investments in agents from $N \setminus \{j\}$ by an amount $\varepsilon_i > 0$ and, at the same time, agent j decreases the sum of her investments in agents from $N \setminus \{i\}$ by an amount $\varepsilon_j > 0$. Note that, if agents i and j are linked in the proposed strategy x, then agent i can decrease her investment in agents $N \setminus \{j\}$ by a positive amount only if $\sum_{k \in N \setminus \{j\}} x_{ik} > 0$. Given the monotonicity considered on the utility function u (Assumption 2-(2)), it follows that, even in the case in which $x_{ij} = 1$ in the proposed strategy x, we still have $\sum_{k \in N \setminus \{j\}} x_{ik} > 0$ for agent i's optimal choice as long as $R_m > 1$. Assumption 3 leads then to that we can always propose the suggested decrease of the sum of agent i's investments in some agents different from agent j. The argument is totally analogous for agent j to be able save an investment $\varepsilon_j > 0$ in her links with agents different from agent i.

(ii) Agent i invests the saved amount ε_i in agent j and, at the same time, agent j invests the saved amount ε_j in agent i.

This class of deviations allows for: (1) if agents $i, j \in N_\theta$, then we obtain new values $s'_i = s_i + \varepsilon_j$ and $s'_j = s_j + \varepsilon_i$ for the total qualities of same-type links, while the total qualities of different-type links d_i and d_j remain unchanged; (2) if $i \in N_\theta$ and $j \in N_{\theta'}$, then we obtain new values $d'_i = d_i + \varepsilon_j$ and $d'_j = d_j + \varepsilon_i$ for the total qualities of different-type links, while the total qualities of same-type links s_i and s_j remain unchanged. Therefore, given that preferences are monotone, is it strictly profitable for both agents i and j to jointly follow this class of deviations. Observe then that this class of deviations is avoided if the strategy profile x does not allow for part (ii) of the deviation to be feasible. Specifically, such deviations are not feasible if, for each pair of different agents in the population, at least one of the agents is already investing with full intensity in the other agent under the strategy profile. Furthermore, in the absence of this condition, Assumption 2 and Assumption 3 ensure that the class of described deviations is the only possible one where, starting from a profile in which all agents exhaust their resources, both agents from a given pair can strictly benefit by deviating bilaterally from such a profile. This is so because, given that preferences are monotone in the investments of the agents and that they exhaust their available resources, any other deviation where a pair of agents do not redirect third-party investments into each other would leave at least one of the agents indifferent.

Finally, we must also verify that the size of the resource R_m allows for at least one agent from each pair of different agents in the population to invest with full intensity in the other agent. Given that the resource R_m is uniform across all agents, notice that if R_m allows each agent i to invest with full intensity $x_{ij} = 1$ in (1) half of the number of remaining agents in the population if n is odd, or in (2) half of the entire number of agents in the population, if n is even, then we can guarantee the condition required by the lemma to prevent the described class of bilateral profitable deviations. Indeed, note that such lower bounds on R_m give us the smallest sizes of the resource that allow for the required condition on the strategy profile x to be satisfied. Since $n_A \geq n/2$ and Assumption 3 requires that $R_m \geq n_A + 1$, we can guarantee that the agents have always sufficient amount of resource R_m to satisfy the condition stated in the lemma. ∎

Proof of *Lemma 3*. Consider a given investment profile x that induces a friendship network $g = g(x)$. Note first that the homophily index of type θ specified in Definition 4 can be rewritten as

$$H_\theta(x) = \frac{\bar{s}_\theta(x)}{\bar{s}_\theta(x) + \bar{d}_\theta(x)} = \frac{1}{1 + \frac{\bar{d}_\theta(x)}{\bar{s}_\theta(x)}}.$$

Then, it follows for our setup with two population groups N_θ and $N_{\theta'}$, that the condition that describes inbreeding homophily satisfies

$$H_\theta(x) > \frac{n_\theta}{n} \quad \Leftrightarrow \quad \frac{\sum_{i \in N_\theta} d_i(x)}{\sum_{i \in N_\theta} s_i(x)} < \frac{n_{\theta'}}{n_\theta}.$$

First, (i) suppose that $g = g(x)$ is a maximally homophilic networks. Then, for each type $\theta \in \Theta$, we have $\sum_{i \in N_\theta} s_i(x) = n_\theta(n_\theta - 1)$ and

$$\begin{aligned}\sum_{i \in N_\theta} d_i(x) &= \sum_{i \in N_\theta} \frac{R_m - (n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ji}}{2} \\ &= \frac{n_\theta[R_m - (n_\theta - 1)] + \sum_{j \in N_{\theta'}} \sum_{i \in N_\theta} x_{ji}}{2} = \frac{nR_m - n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1)}{2}.\end{aligned}$$

Thus,

$$\frac{\sum_{i \in N_\theta} d_i(x)}{\sum_{i \in N_\theta} s_i(x)} = \frac{nR_m - n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1)}{2n_\theta(n_\theta - 1)}.$$

It follows then that

$$\begin{aligned} H_\theta(x) > \frac{n_\theta}{n} &\Leftrightarrow \frac{nR_m - n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1)}{2n_\theta(n_\theta - 1)} < \frac{n_{\theta'}}{n_\theta} \\ &\Leftrightarrow nR_m < 2n_\theta n_{\theta'} - 2n_{\theta'} + n_\theta^2 - n_\theta + (n_{\theta'})^2 - n_{\theta'} \\ &\Leftrightarrow nR_m < \left(n_\theta + n'_\theta\right)^2 - 3(n - n_\theta) - n_\theta \\ &\Leftrightarrow nR_m < n^2 - 3n + 2n_\theta \\ &\Leftrightarrow R_m < (n - 3) + 2n_\theta/n \\ &\Leftrightarrow m < (n_B - 3) + 2n_\theta/(n_A + n_B). \end{aligned}$$

Secondly, (ii) suppose that $g = g(x)$ is a minimally homophilic networks. Then, for each type $\theta \in \Theta$, we have $\sum_{i \in N_\theta} d_i(x) = n_\theta n_{\theta'}$ and

$$\sum_{i \in N_\theta} s_i(x) = \sum_{i \in N_\theta} \frac{[R_m - n_{\theta'}] + \sum_{j \in N^i_\theta} x_{ji}}{2}$$
$$= \frac{n_\theta [R_m - n_{\theta'}] + \sum_{j \in N^i_\theta} \sum_{i \in N_\theta} x_{ji}}{2} = n_\theta [R_m - n_{\theta'}].$$

We can then derive

$$\frac{\sum_{i \in N_\theta} d_i(x)}{\sum_{i \in N_\theta} s_i(x)} = \frac{n_{\theta'}}{R_m - n_{\theta'}}.$$

Therefore,

$$H_\theta(x) < \frac{n_\theta}{n} \iff \frac{n_{\theta'}}{R_m - n_{\theta'}} > \frac{n_{\theta'}}{n_\theta} \iff R_m < n.$$

Since $R_m \leq n-2$ by Assumption 3, it follows then that each minimally homophilic network satisfies inbreeding heterophily with respect to each type $\theta \in \Theta$. ∎

Proof of *Proposition 1*. Consider a strategy profile x that induces a maximally homophilic network $g = g(x)$. Hence, for each agent $i \in N$ of type θ, we have $x_{ij} = 1$ for each $j \in N^i_\theta$ and $\sum_{j \in N_{\theta'}} x_{ij} = R_m - (n_\theta - 1)$ for the type $\theta' \neq \theta$.

1. *Robustness against unilateral deviations:* First, it directly follows that $I^s_i(x_{-i}) = (1/2)(n_\theta - 1)$ for each agent i of type θ. Then, the particular value $\underline{\beta}(\theta; x_{-i})$ of the slope β specified in Eq. (5), under which each agent i of type θ is indifferent between investing with full intensity in links to each other agent of her same type and investing less, equals:

$$\underline{\beta}(\theta; x_{-i}) = \frac{R_m - (n_\theta - 1) + 2I^d_i(x_{-i})}{2(n_\theta - 1)}.$$

Therefore, if for each possible type $\theta \in \Theta$, and each type $\theta' \neq \theta$, the level β of assortative interests equals the indifference cutoff value $\underline{\beta}(\theta; x_{-i})$ above, then no agent has unilateral incentives to deviate from the proposed strategy profile x, as stated by condition 1. of the proposition. On the other hand, if $\beta > \underline{\beta}(\theta; x_{-i})$, then such an agent $i \in N_\theta$ has incentives to deviate from investing with full intensity in each other agent of her same type. Thus, the inequality $\beta \leq \underline{\beta}(\theta; x_{-i})$, for each $i \in N_\theta$ and each type $\theta \in \Theta$, gives us a necessary condition for x to be stable.

2. *Robustness against bilateral deviations:* Note first that if $\beta \leq \underline{\beta}(\theta; x_{-i})$, then no agent of type θ has incentives to lower her full-intensity investments in each other agent of type θ. Therefore, no pair of two different agents of the same type θ have incentives to deviate from investing with full intensity in each other agent of type θ either. The only possible class of profitable bilateral deviations that remains to be ruled out must then involve two agents of different types. In particular, since $n_\theta \geq 3$ for each type θ, we can

consider a deviation by a pair of agents i and j, with $i \in N_A$ and $j \in N_B$, in which each of the two agents redirect third-party investments into each other. Then, as already argued in the proof of Lemma 2, such a (unique) class of bilateral deviations is prevented if for each pair of agents that belong to different groups at least one of the agents is already investing with full intensity in the other agent, as stated in 2. of the proposition.

Finally, we must also verify that the size of the resource R_m allows for the type of connections described in conditions 1. and 2. of the proposition to be feasible for all agents in the population. Note first, that under a strategy profile x that induces a maximally homophilic network, the capacity constraint requirement (Assumption 3) for each agent $i \in N_\theta$, for $\theta \in \Theta$, takes the form

$$(n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ij} \leq R_m.$$

By aggregating the requirement above across all agents $i \in N_\theta$, for both types $\theta \in \Theta$, it follows then that the size of the resource R_m must necessarily satisfy

$$n_A(n_A - 1) + n_B(n_B - 1) + \sum_{i \in N_A} \sum_{j \in N_B} x_{ij} + \sum_{i \in N_B} \sum_{j \in N_A} x_{ij} \leq nR_m. \tag{10}$$

Note that the number of possible pairs $(i, j) \in N_A \times N_B$ of different-type agents is $n_A n_B$. In addition, if for each of such $n_A n_B$ different possible pairs, we require that at least one of the agents from the pair invests with full intensity in the other agent, then the minimum aggregate quality for the connections among different-type agents required to satisfy condition 2. amounts precisely to $n_A n_B$. Therefore, any profile x that satisfies condition 2. of the proposition must necessarily satisfy $n_A n_B \leq \sum_{i \in N_A} \sum_{j \in N_B} x_{ij} + \sum_{i \in N_B} \sum_{j \in N_A} x_{ij}$. Then, by combining this equality with the condition in Eq. (10) above, we obtain that

$$n_A(n_A - 1) + n_B(n_B - 1) + n_A n_B \leq nR_m$$

follows as a necessary requirement from condition 2 of the proposition. Simple algebra allows to rewrite the inequality above as $R_m \geq (n-1) - n_A n_B / n$, as stated in condition 2. of the proposition. This concludes all the required arguments. ∎

Proof of *Corollary 2*. The sufficient conditions for β and R_m derived by Corollary 2 follow from the requirements of Proposition 1. First, note that that, for the class of strategy profiles x proposed in the corollary, we have $I_i^d(x_{-i}) \geq (n_B - 1)/2$ for each agent $i \in N_\theta$ and each type $\theta \in \Theta$. Then, by combining the lower bound $(n_B - 1)/2$ on the total incoming intensity $I_i^d(x_{-i})$ with the condition 1. derived in Proposition 1, it follows that $\beta \leq [R_m + (n_B - n_A)]/2(n_A - 1)$ is a sufficient condition for all agents to have incentives to invest with full intensity in each other same-type agent. Secondly, note that condition 2. of Proposition 1 is satisfied by construction for the strategy profiles x described by the corollary. Thirdly, consider an agent $i \in N_\theta$, for $\theta \in \Theta$, who makes investments as prescribed by the class of strategy profiles x proposed in Corollary 2 but does not invest any extra amount on any other different-type agent. Then, it follows that x_i satisfies

$$\sum_{j \in N_\theta^i} x_{ij} + \sum_{j \in N_{\theta'}} x_{ij} = (n_\theta - 1) + n_{\theta'}/2 \quad \text{for } n_{\theta'} \text{ even;}$$
$$\sum_{j \in N_\theta^i} x_{ij} + \sum_{j \in N_{\theta'}} x_{ij} = (n_\theta - 1) + (n_{\theta'} - 1)/2 + 1 \quad \text{for } n_{\theta'} \text{ odd.}$$

In addition, we know that

$$(n_\theta - 1) + (n_{\theta'} - 1)/2 + 1 \leq n_A + (n_B - 1)/2$$

for each agent $i \in N_\theta$ and each type $\theta \in \Theta$. Therefore, if $R_m \geq n_A + (n_B - 1)/2$, we can then ensure that each agent has the amount of resource R_m required to follow the prescription for the class of strategy profiles x proposed by Corollary 2. ∎

Proof of *Proposition 2*. Consider a strategy profile x that induces a minimally homophilic network $g = g(x)$. Hence, for each agent $i \in N$ of type θ, we have $x_{ij} = 1$ for each $j \in N_{\theta'}$ and $\sum_{j \in N_\theta^i} x_{ij} = R_m - n_{\theta'}$ for the type $\theta' \neq \theta$.

1. *Robustness against unilateral deviations:* It follows directly that $I_i^d(x_{-i}) = (1/2)n_{\theta'}$ for each agent i of type θ. Then, the particular value $\overline{\beta}(\theta; x_{-i})$ of the slope β specified in Eq. (5), under which each agent i of type θ is indifferent between investing with full intensity in links to each different-type agent and investing less, equals:

$$\overline{\beta}(\theta; x_{-i}) = \frac{2n_{\theta'}}{(R_m - n_{\theta'}) + 2I_i^s(x_{-i})}.$$

Therefore, if for each possible type $\theta \in \Theta$, and each type $\theta' \neq \theta$, the level β of assortative interests equals the indifference value $\overline{\beta}(\theta; x_{-i})$ above, then no agent has unilateral incentives to deviate from the proposed strategy profile x, as stated by condition 1. of the proposition. On the other hand, if $\beta < \overline{\beta}(\theta; x_{-i})$, then such an agent $i \in N_\theta$ has incentives to deviate from investing with full intensity in each different-type agent. Thus, the inequality $\beta \geq \overline{\beta}(\theta; x_{-i})$, for each $i \in N_\theta$ and each type $\theta \in \Theta$, gives us a necessary condition for x to be stable.

2. *Robustness against bilateral deviations:* Note first that if $\beta \geq \overline{\beta}(\theta; x_{-i})$, then no agent of type θ has incentives to lower her full-intensity investments in each agent of type θ'. Therefore, no pair of two different agents have incentives to deviate from investing with full intensity in each other either. The only possible class of profitable bilateral deviations that remains to be ruled out must then involve two agents of the same type. In particular, since $n_\theta \geq 3$ for each type $\theta \in \Theta$, we can consider a deviation by a pair of agents $i, j \in N_\theta$, for $i \neq j$, in which both agents redirect third-party investments into each other. Then, as already argued in the proof of Lemma 2, such a (unique) class of bilateral deviations is

prevented if for each pair of agents that belong to the same group, at least one of the agents is already investing with full intensity in the other agent, as stated in 2. of the proposition.

Finally, we must also verify that the size of the resource R_m allows for the type of connections described in conditions 1. and 2. of the proposition to be feasible for all agents in the population. Note first, that under a strategy profile x that induces a minimally homophilic network, the capacity constraint requirement (Assumption 3) for each agent $i \in N_\theta$, for $\theta \in \Theta$, takes the form

$$\sum_{j \in N_\theta \setminus \{j\}} x_{ij} + n_{\theta'} \leq R_m.$$

By aggregating the requirement above across all agents $i \in N_\theta$, for both types $\theta \in \Theta$, it follows then that the size of the resource R_m must necessarily satisfy

$$\sum_{i \in N_\theta} \sum_{j \in N_\theta \setminus \{j\}} x_{ij} + n_\theta n_{\theta'} \leq n_\theta R_m \tag{11}$$

Note that the number of possible pairs $(i, j) \in N_\theta \times N_\theta$, with $i \neq j$, between same-type agents is $n_\theta(n_\theta - 1)$. Then, if for each of such $n_\theta(n_\theta - 1)$ different possible pairs, we require that at least one of the agents from the pair invests with full intensity in the other agent, it follows that the aggregate quality between all the agents of type θ, for any profile x that satisfies condition 2. of the proposition, must be at least $n_\theta(n_\theta - 1)/2$ Therefore, a minimally homophilic network that satisfies condition 2. of the proposition must necessarily satisfy $n_\theta(n_\theta - 1)/2 \leq \sum_{i \in N_\theta} \sum_{j \in N_\theta \setminus \{j\}} x_{ij}$. Then, by combining this equality with the condition in Eq. (11) above, we obtain that $(n_\theta - 1)/2 + n_{\theta'} \leq R_m$ for each $\theta \in \Theta$ is a necessary requirement for condition 2. of the proposition to be satisfied. Since we are considering $n_A \geq n_B$ without loss of generality, it follows that $R_m \geq n_A + (n_B - 1)/2$ is the required necessary condition on the size of the total resource. This concludes all the required arguments. ∎

Proof of *Corollary 4*. The sufficient conditions for β and R_m derived by Corollary 4 follow from the requirements of Proposition 2. First, note that that, for the class of strategy profiles x proposed in the corollary, we have $I_i^s(x_{-i})$ is always lower for agents $i \in N_B$ than for agents $i \in N_A$. Also, for each agent $i \in N_B$, we have $I_i^s(x_{-i}) = m$. Then, by combining the lower bound m on the total incoming intensity $I_i^s(x_{-i})$ with the condition 1. derived in Proposition 2, it follows that $\beta \geq 2n_A/(R_m - n_A + m)$ is a sufficient condition for all agents to have incentives to invest with full intensity in each different-type agent. Secondly, note that condition 2. of Proposition 2 is satisfied by construction for the strategy profiles x described by the corollary. Secondly, it is easy to verify that, by construction, the proposed strategy profile always satisfies the key condition given in Lemma 2 to prevent profitable bilateral deviations. Finally, since we are considering that $n_A \geq n_B$, it follows that $l_A = n_A - n_B$ for each $i \in N_A$, whereas $l_B = 0$ for each $i \in N_B$ Therefore, if $R_m \geq n_A + \alpha(n_B)$, then we can ensure that each agent has, at least, the amount R_m of the resource required to follow the prescription for the class of strategy profiles x proposed by Corollary 4. ∎

Proof of Proposition 3. We prove both statements (i) and (ii) of the proposition by contradiction.

(i) Consider a strategy profile x that induces a maximally homophilic network $g = g(x)$. Then, for each agent $i \in N_\theta$, and each type $\theta \in \Theta$, we have $x_{ij} = 1$ for each $j \in N_\theta^i$. Therefore, $I_i^s(x_{-i}) = (1/2)(n_\theta - 1)$ for each agent $i \in N_\theta$, and each type $\theta \in \Theta$. Then, using the expression of the upper bound $\overline{\beta}(\theta; x_{-i})$ for the indifference value of β, which is associated to the unilateral optimal choice described by [a] in Fig. 1, it follows that

$$\hat{\beta}_{i,\theta}(x) \equiv \frac{R_m - (n_\theta - 1) + 2I_i^d(x_{-i})}{2(n_\theta - 1)} \tag{12}$$

gives us the indifference value for the level of assortative interests under which agent i is indifferent between investing with full intensity in each other same-type agent and investing lower amounts in some same-type agent. First, suppose that the strategy profile x is such that $x_{ji} = [R_m - (n_{\theta'} - 1)]/n_\theta$ for each pair of agents $i \in N_\theta$, and $j \in N_{\theta'}$, for each type $\theta \in \Theta$ and the type $\theta' \neq \theta$. Thus, we are considering a strategy profile x where each agent in the population receives a constant proportional amount of investments from each different-type agent. In this case the investment received by each agent from each different-type agent depends only on the group to which she belongs. In particular, for each $i \in N_\theta$ and each $\theta \in \Theta$, the indifference value specified in Eq. (12) takes the form

$$\hat{\beta}_\theta \equiv \frac{nR_m - n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1)}{2(n_\theta - 1)n_\theta}.$$

Suppose that $\beta \in (1, +\infty)$. Then, each agent $i \in N_\theta$ has (weak) incentives to invest with full intensity in each other same-type agent only if

$$\hat{\beta}_\theta > 1 \;\Leftrightarrow\; nR_m - 3n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1) > 0.$$

Now, recall that the capacity constraint described by Assumption 3 imposes that $R_m < n_\theta + n_{\theta'} - 1$. Therefore, we know that

$$\begin{aligned} nR_m - 3n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1) &< (n_\theta + n_{\theta'})(n_\theta + n_{\theta'} - 1) - 3n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1) \\ &= 2(n'_\theta - n_\theta + 1) < 0 \;\text{ for some type } \theta \in \Theta, \text{ and for } \theta' \neq \theta, \end{aligned}$$

since we are considering that $n_A \geq n_B$. Therefore, each agent $i \in N_A$ has (strict) incentives to deviate from the proposed profile x that induces a maximally homophilic network. Secondly, consider another strategy profile $x' \neq x$ that induces as well a maximally homophilic network $g = g(x')$ and such that $\tilde{\beta}_{i,\theta}(x') > \tilde{\beta}_\theta$. Recall that preferences are monotone and, therefore, that the resource constraint $\sum_{j \neq i} x'_{ij} \leq R_m$ must be satisfied with equality for each agent who has no unilateral incentives to deviate from x'. Then, it must be the case that $\tilde{\beta}_{j,\theta}(x') < \tilde{\beta}_{\theta}$ for some other agent $j \in N_{\theta}$. In other words, if $\tilde{\beta}_{i,\theta}(x') > 1$ for some agent $i \in N_{\theta}$, then it must be the case that $\tilde{\beta}_{j,\theta}(x') < 1$ for some other agent $j \in N_{\theta}$. In this case, we observe that such an agent j would have (strict) incentives to deviate unilaterally from x'. Therefore, we conclude that if $\beta \in (1, +\infty)$, then at least one agent in the population has unilateral incentives to deviate from any profile that induces a maximally homophilic network.

(ii) Consider a strategy profile x that induces a minimally homophilic network $g = g(x)$. Then, for each agent $i \in N_{\theta}$, and each type $\theta \in \Theta$, we have $x_{ij} = 1$ for each $j \in N_{\theta'}$ for the type $\theta \neq \theta$. Therefore, $I_i^d(x_{-i}) = (1/2)n_{\theta'}$ for each agent $i \in N_{\theta}$, and each type $\theta \in \Theta$. Then, using the expression of the upper bound $\overline{\beta}(\theta; x_{-i})$ for the indifference value of β, which is associated to the unilateral optimal choice described by [c] in Fig. 1, it follows that

$$\tilde{\beta}_{i,\theta}(x) \equiv \frac{2n_{\theta'}}{(R_m - n_{\theta'}) + 2I_i^s(x_{-i})} \tag{13}$$

gives us the indifference value for the level of assortative interests under which agent i is indifferent between investing with full intensity in each different-type agent and investing lower amounts in some different-type agent. First, suppose that the strategy profile x is such that $x_{ji} = (R_m - n_{\theta'})/(n_{\theta} - 1)$ for each pair of agent $i, j \in N_{\theta}$, with $i \neq j$, and for each type $\theta \in \Theta$. Thus, we are considering a strategy profile x where each agent in the population receives a constant proportional amount of investments from each other same-type agent. In this case, the investment received by each agent from each other same-type agent depends only on the group to which she belongs. In particular, for each $i \in N_{\theta}$ and each $\theta \in \Theta$, the indifference value specified in Eq. (13) takes the form

$$\tilde{\beta}_{\theta} \equiv \frac{n_{\theta'}}{(R_m - n_{\theta'})}.$$

Suppose that $\beta \in (0, 1]$. Then, each agent $i \in N_{\theta}$ has (weak) incentives to invest with full intensity in each different-type agent only if $\tilde{\beta}_{\theta} \leq 1$. Using the expression for $\tilde{\beta}_{\theta}$ derived above, we observe that this is possible for each type $\theta \in \Theta$ only if $R_m > 2n_A$ and $R_m > 2n_B$ simultaneously. However, this leads to a contradiction since such inequalities cannot happen simultaneously given that Assumption 3 requires that $R_m < n - 1$. Secondly, consider another strategy profile $x' \neq x$ that induces as well a minimally homophilic network $g = g(x')$ and such that $\tilde{\beta}_{i,\theta}(x') < \tilde{\beta}_{\theta}$. Recall that preferences are monotone and, therefore, that the resource constraint $\sum_{j \neq i} x'_{ij} \leq R_m$ (described by Assumption 3) must be satisfied with equality for each agent who has no unilateral incentives to deviate from x'. Then, it must be the case that $\tilde{\beta}_{j,\theta}(x') > \tilde{\beta}_{\theta}$ for some other agent $j \in N_{\theta}$. In other words, if $\tilde{\beta}_{i,\theta}(x') \leq 1$ for some agent $i \in N_{\theta}$, then it must be the case that $\tilde{\beta}_{j,\theta}(x') > 1$ for some other agent $j \in N_{\theta}$. In this case, we observe that such an agent j would have strict incentives to deviate unilaterally from x'. Therefore, we conclude that if $\beta \in (0, 1]$, then at least one agent in the population has unilateral incentives to deviate from any profile that induces a minimally homophilic network. ∎

Proof of *Corollary 5*. First, consider a strategy profile x that induces a maximally homophilic network $g = g(x)$. Stability of such a network requires that no agent wants to deviate unilaterally from the proposed strategy profile x. Specifically, stability of a maximally homophilic network $g = g(x)$ requires that

$$\beta \leq \hat{\beta}(x) = \inf_{i \in N_\theta, \theta \in \Theta} \frac{R_m - (n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ji}}{2(n_\theta - 1)}.$$

Now, let us complete the description of the proposed profile x by requiring that each agent $i \in N_\theta$, for each $\theta \in \Theta$, receives a common intensity of investments from the different-type agents. Thus, consider that x satisfies $\sum_{j \in N_{\theta'}} x_{ji} = n_{\theta'}[R_m - (n_{\theta'} - 1)]/n_\theta$ for each $i \in N_\theta$ and each $\theta \in \Theta$. This construction of x entails that the highest possible cutoff value $\hat{\beta}(x)$ for the optimal unilateral behavior where agents want to invest with full intensity in all other same-type agents (which was described by [a] of Fig. 1) cannot exceed one. Then, given that $n_A \geq n_B$, we observe that such a proposed maximally homophilic network $g = g(x)$ satisfies the criterion of robustness against unilateral deviations if and only if

$$\begin{aligned} \beta \leq & \frac{nR_m - [n_A(n_A - 1)) + n_B(n_B - 1)]}{2n_A(n_A - 1)} \\ & = \frac{n(m+1) + n_B(n_A - n_B)}{2n_A(n_A - 1)} \equiv \beta_l(m), \end{aligned}$$

where, as derived in the proof of Proposition 3 (i), the cutoff value $\beta_l(m)$ in the right-hand side of the expression above cannot exceed one. Furthermore, Proposition 3 established that if $\beta \leq 1$, then there do not exist stable minimally homophilic networks. As a consequence, provided that the corresponding cutoff value $\beta_l(m)$ is strictly less than one, if $\beta \in (\beta_l(m), 1]$, then all stable networks must necessarily be partially homophilic . For the case in which the cutoff value $\beta_l(m)$ equals one, recall that Proposition 3 guaranteed then that stable maximally homophilic networks do not exist for $\beta > 1$.

Secondly, consider a strategy profile x that induces a minimally homophilic network $g = g(x)$. Recall that robustness against unilateral deviations for such a network to be stable it requires that

$$\beta \geq \tilde{\beta}(x) = \sup_{i \in N_\theta, \theta \in \Theta} \frac{2n_{\theta'}}{(R_m - n_{\theta'}) + \sum_{j \in N_i^\theta} x_{ji}}.$$

Let us complete the description of the proposed profile x by requiring that each agent $i \in N_\theta$, for each $\theta \in \Theta$, receives a common intensity of investments from the same-type agents. Thus, consider that x satisfies $\sum_{j \in N_i^\theta} x_{ji} = R_m - n_{\theta'}$ for each $i \in N_\theta$ and each $\theta \in \Theta$. This proposal gives us a profile x that yields the lowest possible cutoff value $\tilde{\beta}(x)$ for the optimal unilateral behavior where all agents want to invest with full intensity in all different-type agents (which was described by [c] of Fig. 1). Since we are considering that $n_A \geq n_B$, it follows that such a proposed minimally homophilic network $g = g(x)$ satisfies the criterion of robustness against unilateral deviations if and only if

$$\beta \geq \frac{n_A}{R_m - n_A} = \frac{n_A}{m} \equiv \beta_h(m),$$

where the cutoff value $\beta_h(m)$ exceeds one. It follows from Proposition 3 (ii), that if $\beta > 1$, then stable maximally homophilic networks do not exist. As a consequence, we know that if $\beta \in (1, \beta_h(m)]$, then all stable networks must necessarily be partially homophilic .

This completes our derivation of an interval $[\beta_l(m), \beta_h(m)]$ of "intermediate" assortative for which only partially homophilic networks are stable friendship networks. ∎

Proof of *Proposition 4*. Consider the social value function v, defined in Eq. (8). First, consider an arbitrary investment profile $x \in X$ that induces a collection of sets of pairs $(\{(s_i(x), d_i(x))\}_{i \in N_A}, \{(s_j(x), d_j(x))\}_{j \in N_B})$ of same-type and different-type qualities. Then, the average of the qualities for same-type and different-type links, respectively, across all agents in each group N_θ can be computed as

$$\bar{s}_\theta(x) = (1/n_\theta) \sum_{i \in N_\theta} s_i(x) = (1/2n_\theta) \sum_{i \in N_\theta} \sum_{j \in N_\theta^i} [x_{ij} + x_{ji}] \tag{14}$$

and

$$\bar{d}_\theta(x) = (1/n_\theta) \sum_{i \in N_\theta} d_i(x) = (1/2n_\theta) \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} [x_{ij} + x_{ji}]. \tag{15}$$

Secondly, using the definition of same-type $s_i(x)$ and different-type $d_i(x)$ aggregate qualities, let us propose another investment profile $\hat{x} \in X$ such that $s_i(\hat{x})$ and $d_i(\hat{x})$ be constant across all agents $i \in N_\theta$ for each type $\theta \in \Theta$. From the definition of the aggregate qualities, it follows that the quantities $\sum_{j \in N_\theta^i} \hat{x}_{ij}$, $\sum_{j \in N_\theta^i} \hat{x}_{ji}$, $\sum_{j \in N_{\theta'}} \hat{x}_{ij}$, and $\sum_{j \in N_{\theta'}} \hat{x}_{ji}$ must be constant across agents within each population group. Accordingly, we start by proposing a profile $\hat{x}$ such that, for each agent $i \in N_\theta$ and each type $\theta \in \Theta$, we have

(a) $\sum_{j \in N_\theta^i} \hat{x}_{ij} = y_{\theta\theta}$ and $\sum_{j \in N_\theta^i} \hat{x}_{ji} = z_{\theta\theta}$, and

(b) $\sum_{j \in N_{\theta'}} \hat{x}_{ij} = y_{\theta\theta'}$ and $\sum_{j \in N_{\theta'}} \hat{x}_{ji} = z_{\theta\theta'}$.

In particular, for any agent $i \in N_\theta$, the amount $y_{\theta\theta}$ describes i's total investments in the rest of her same-type agents, whereas $y_{\theta\theta'}$ describes i's aggregate investments in all different-type agents. Similarly, for any agent $i \in N_\theta$, the amount $z_{\theta\theta}$ describes the total of investments that i receives from the rest of her same-type agents, whereas $z_{\theta\theta'}$ describes i's aggregate investments that i receives from all different-type agents. Thus, under the profile $\hat{x}$, the sums of the aggregate outgoing and incoming investments are only contingent on the characteristics of the agents.

Given the proposal above, note first that, by summing the investments made and received over all same-type agents for any type, it follows that $n_\theta y_{\theta\theta} = n_\theta z_{\theta\theta}$, so that it must necessarily be the case that $y_{\theta\theta} = z_{\theta\theta}$. Secondly, by noting that the sum of the investments made by all agents from N_θ in all the agents of the group $N_{\theta'}$ must be equal to the sum of the investments received by all agents of the group $N_{\theta'}$ from all agents from N_θ, it follows $n_\theta y_{\theta\theta'} = n_{\theta'} z_{\theta'\theta}$. Our proposal accordingly incorporates also these two considerations. Crucially, from the definitions of s_i and d_i, it follows that the characteristics imposed by our proposal for the profile $\hat{x}$ are necessary and sufficient to make $s_i(\hat{x}) = s_j(\hat{x})$ and $d_i(\hat{x}) = d_j(\hat{x})$ for each pair of (distinct) agents $i, j \in N_\theta$, for each type $\theta \in \Theta$.

Furthermore, consider that, for such a profile $\hat{x}$, each agent satisfies her capacity constraint (Assumption 3) with equality. Then, the constant investments proposed by means of $\hat{x}$ must satisfy

$$y_{\theta\theta} + y_{\theta\theta'} = R_m \quad \text{for each } \theta \in \Theta, \quad \text{and for } \theta' \neq \theta. \tag{16}$$

The associated qualities are simply derived as $s_i(\hat{x}) = (1/2)[y_{\theta\theta} + z_{\theta\theta}] = y_{\theta\theta}$ and $d_i(\hat{x}) = (1/2)[y_{\theta\theta'} + z_{\theta\theta'}]$, where, as indicated above, we must also consider that $z_{\theta\theta'} = (n_{\theta'}/n_\theta) y_{\theta'\theta}$, for each agent $i \in N_\theta$, each type $\theta \in \Theta$, and $\theta' \neq \theta$.

Now, we can set a relationship between the linkage qualities associated to $\hat{x}$, which are constant across all agents within each population group, and the average qualities derived in Eq. (14) and Eq. (15) for the profile x. By requiring $s_i(\hat{x}) = \bar{s}_\theta(x)$ and $d_i(\hat{x}) = \bar{d}_\theta(x)$ for each $i \in N_\theta$ and each $\theta \in \Theta$, we obtain

$$\begin{aligned} y_{\theta\theta} &= \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_\theta^i} x_{ij}, \quad z_{\theta\theta} = \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_\theta^i} x_{ji}, \quad \text{and} \\ y_{\theta\theta'} &= \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} x_{ij}, \quad z_{\theta\theta'} = \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} x_{ji}. \end{aligned} \tag{17}$$

Conditional on the above established relationship (Eq. (17)) between the profiles x and $\hat{x}$, clearly the profile $\hat{x}$ satisfies the capacity condition required by Eq. (16):

$$y_{\theta\theta} + y_{\theta\theta'} = (1/n_\theta)\Big(\sum_{i \in N_\theta} \sum_{j \in N_\theta^i} x_{ij} + \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} x_{ij} \Big) = R_m.$$

Also, as required, notice that our relationship between the two investment profiles, entails that $\hat{x}$ satisfies $\hat{x}_{ij} \in [0, 1]$ for each pair of (distinct) agents $i, j \in N$.

Therefore, we are able to establish the key equality $u(s_i(\hat{x}), d_i(\hat{x})) = u(\bar{s}_\theta(x), \bar{d}_\theta(x))$ for each agent $i \in N_\theta$, and each $\theta \in \Theta$, where each $(s_i(\hat{x}), d_i(\hat{x})) \in D_i(\hat{x}_{-i})$ for each agent i. Importantly, we can establish such an equality regardless of whether $(\bar{s}_\theta(x), \bar{d}_\theta(x))$ belongs to the feasible set $D_i(x_{-i})$ for each agent $i \in N_\theta$, and each $\theta \in \Theta$. Now, since the utility function u is (strictly) concave in the (s_i, d_i) space (Assumption 2–(3)), it follows that

$$\begin{aligned} v(g(\hat{x})) &= \sum_{i\in N} u(s_i(\hat{x}), d_i(\hat{x})) = \sum_{\theta\in\Theta}\sum_{i\in N_\theta} u(\bar{s}_\theta(x), \bar{d}_\theta(x)) \\ &= \sum_{\theta\in\Theta}\sum_{i\in N_\theta} u\Big((1/n_\theta)\sum_{i\in N_\theta} s_i(x), (1/n_\theta)\sum_{i\in N_\theta} d_i(x)\Big) \\ &\geq \sum_{\theta\in\Theta}\sum_{i\in N_\theta}(1/n_\theta)\sum_{i\in N_\theta} u(s_i(x), d_i(x)) = \sum_{\theta\in\Theta}\sum_{i\in N_\theta} u(s_i(x), d_i(x)) = v(g(x)), \end{aligned}$$

where the inequality above holds strictly unless our initial investment profile x satisfies $s_i(x) = \bar{s}_\theta(x)$ and $d_i(x) = \bar{d}_\theta(x)$ for each $i \in N_\theta$ and each $\theta \in \Theta$. It follows then that an efficient network $g = g(\hat{x})$ requires that the qualities $(s_i(\hat{x}), d_i(\hat{x}))$ be constant across all agents i within each of the two population groups. ∎

Proof of *Proposition 5*. Let $\hat{x}$ be a strategy profile that satisfies the necessary condition given by Proposition 4. Then, we can fully describe the profile $\hat{x}$ using the type-contingent aggregate investments y_{AA}, y_{BB}. The social planner can select in a totally independent way the pair of variables y_{AA}, y_{BB}, under the respective restrictions $y_{AA} \in [R_m - n_B, n_A - 1]$ and $y_{BB} \in [R_m - n_A, n_B - 1]$. In turn, the aggregated investments y_{AB}, y_{BA}, z_{AB}, and z_{BA} can be derived from the optimally selected quantities y_{AA}, y_{BB}. Using the expression of the social value in Eq. (9), the problem that the social planner can thus be set as

$$\begin{aligned} \max_{\{y_{AA}, y_{BB}\}} \; & n_A u\Big(y_{AA}, \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_A}\Big) + n_B u\Big(y_{BB}, \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_B}\Big) \\ \text{s.t.:} \; & y_{AA} \in [R_m - n_B, n_A - 1]; \\ & y_{BB} \in [R_m - n_A, n_B - 1]. \end{aligned} \tag{18}$$

Observation of the problem in Eq. (18) above allows us to proceed as follows.

(i) We identify a sufficient condition on the level of assortative interests β under which, regardless of the aggregate investment choice y_{AA} of the agents from the larger group N_A, the utility of any agent from the smaller group N_B is maximized when she invests with full intensity in all other same-type agents, i.e., $y_{BB} = n_B - 1$. Furthermore, the identified condition on β simultaneously ensures that the agents from the larger group N_A maximize their utilities when they choose to invest with full intensity in all other same-type agents, i.e., $y_{AA} = n_A - 1$, independently of the choice y_{BB} of the agents from the smaller group. Since the welfare function $v(g(\hat{x}))$ aggregates the utilities of all the agents, for each of the two groups, it follows that the derived condition is sufficient to guarantee that the value function is maximized when all agents invest with full intensity in all other same-type agents, i.e., $y_{AA} = n_A - 1$ and $y_{BB} = n_B - 1$.

On the one hand, let us take as given an arbitrary quantity $y_{AA} \in [R_m - n_B, n_A - 1]$, and suppose then that the social planner chooses the quantity y_{BB} in order to maximize the utility of a representative agent of the smaller group, N_B. Thus, we are now restricting attention to the (hypothetical) problem

$$\max_{y_{BB}\in[R_m-n_A,n_B-1]} u\left(y_{BB}, \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_B}\right).$$

Then, recall that Assumption 2–(4) (b) establishes that $\partial u(s_i, d_i)/\partial s_i > \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i > \beta$. Therefore, if

$$\beta < \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_B y_{BB}}$$

for each $y_{BB} \in [R_m - n_A, n_B - 1]$, then we can guarantee that maximization of the utility of any agent $i \in N_B$ is uniquely achieved by selecting $y_{BB} = n_B - 1$, for each possible $y_{AA} \in [R_m - n_B, n_A - 1]$. Furthermore, since the function $[nR_m - n_A y_{AA} - n_B y_{BB}]/2n_B y_{BB}$ is strictly decreasing in y_{BB}, it follows that

$$\beta < \frac{nR_m - n_A y_{AA} - n_B(n_B - 1)}{2n_B(n_B - 1)} \tag{19}$$

is a sufficient condition that ensures maximization of the utility of the agents $j \in N_B$ is characterized by $y_{BB} = n_B - 1$, for any given $y_{AA} \in [R_m - n_B, n_A - 1]$.

On the other hand, let us now take as given an arbitrary quantity $y_{BB} \in [R_m-n_A, n_B-1]$, and then restrict attention to the (hypothetical) problem of choosing the value of y_{AA} that solves

$$\max_{y_{AA}\in[R_m-n_B,n_A-1]} u\left(y_{AA}, \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_A}\right).$$

Using again Assumption 2–(4) (b), we can guarantee the solution to the problem above is characterized by $y_{AA} = n_A - 1$ if

$$\beta < \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_A y_{AA}}$$

for each $y_{AA} \in [R_m - n_B, n_A - 1]$. Since the function $[nR_m - n_A y_{AA} - n_B y_{BB}]/2n_A y_{AA}$ is strictly decreasing in y_{AA}, it follows that

$$\beta < \frac{nR_m - n_A(n_A - 1) - n_B y_{BB}}{2n_A(n_A - 1)} \tag{20}$$

is a sufficient condition that ensures maximization of the utility of the agents $i \in N_A$ is characterized by $y_{AA} = n_A - 1$, for any choice $y_{BB} \in [R_m - n_A, n_B - 1]$.

Therefore, if both conditions Eq. (19) and Eq. (20) are simultaneously satisfied for values $y_{AA} = n_A - 1$ and $y_{BB} = n_B - 1$, then the utility of the agents from the smaller group N_B is maximized when they choose $y_{BB} = n_B - 1$ conditional on the choice $y_{AA} = n_A - 1$

while, at the same time, the utility of the agents from the larger group N_A is maximized when they choose $y_{AA} = n_A - 1$ conditional on the choice $y_{BB} = n_B - 1$. Since conditions Eq. (19) and Eq. (20) combined guarantee such (common) features for the optimal choices of the two separate (hypothetical) problems—relative to each of the two populations—, we obtain that such sufficient conditions combined ensure that the only solution to the problem in Eq. (18) entails $y_{AA} = n_A - 1$ and $y_{BB} = n_B - 1$.

Since $n_A \geq n_B$, we have that

$$\frac{nR_m - n_A(n_A - 1) - n_B(n_B - 1)}{2n_B(n_B - 1)} \geq \frac{nR_m - n_A(n_A - 1) - n_B(n_B - 1)}{2n_A(n_A - 1)}.$$

In addition, recall from Eq. (3) the expression of the particular value

$$\begin{aligned}\beta_l(m) &= [nR_m - n_A(n_A - 1) - n_B(n_B - 1)]/2n_A(n_A - 1)\\ &= [n(m + 1) + n_B(n_A - n_B)]/2n_A(n_A - 1)\end{aligned}$$

for the level of assortative interests. Therefore, if $\beta < \beta_l(m)$, then the only way in which the social planner can maximize the social value $v(g(\hat{x}))$ is by choosing $y_{AA} = n_A - 1$, $y_{BB} = n_B - 1$. Such choices also yield $y_{AB} = m + 1$, $y_{BA} = (m + 1) + (n_A - n_B)$, $z_{AB} = (n_B/n_A)[(m + 1) + (n_A - n_B)]$, and $z_{BA} = (n_A/n_B)(m + 1)$. Accordingly, for each agent $i \in N_\theta$, each type $\theta \in \Theta$, and $\theta' \neq \theta$, an efficient network $\hat{g} = g(\hat{x})$ entails

$$s_i(\hat{x}) = n_\theta - 1 \quad \text{and} \quad d_i(\hat{x}) = \frac{n(m + 1) + n_B(n_A - n_B)}{2n_\theta}.$$

(ii) Similarly to the arguments used in (i), we consider separately two hypothetical problems that address the maximization of the utility of any agent from a given group, regardless of the choices made by the agents from the other group. Again, we derive a sufficient condition on the level of assortative interests β under which, regardless of the aggregate investment choice y_{AA} of the agents from the larger group N_A, the utility of any agent from the smaller group N_B is maximized when the agents invests with full intensity in all different-type agents, i.e., $y_{BB} = R_m - n_A$. Furthermore, such a condition on β guarantees at the same time that the agents from the larger group N_A maximize their utilities when they invest with full intensity in all different-type agents as well, i.e., $y_{AA} = R_n - n_B$, independently of the choice y_{BB} of the agents from the smaller group. The additive nature of the welfare function $v(g(\hat{x}))$ leads then to that the derived condition is sufficient to guarantee that the value function is maximized when all agents invest with full intensity in all other same-type agents, i.e., $y_{AA} = R_n - n_B$ and $y_{BB} = R_n - n_A$.

First, fix an arbitrary quantity $y_{AA} \in [R_m - n_B, n_A - 1]$, and let us look for the quantity y_{BB} that solves the (hypothetical) problem

$$\max_{y_{BB} \in [R_m - n_A, n_B - 1]} u\left(y_{BB}, \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_B}\right).$$

Then, recall Assumption 2–(4) (c) establishes that $\partial u(s_i, d_i)/\partial s_i < \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i < \beta$. Therefore, if

$$\beta > \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_B y_{BB}}$$

for each $y_{BB} \in [R_m - n_A, n_B - 1]$, then we can guarantee that the solution to the problem of this first step is uniquely given by $y_{BB} = R_m - n_A$. Since the function $[nR_m - n_A y_{AA} - n_B y_{BB}]/2n_B y_{BB}$ is strictly decreasing in y_{BB}, it follows that

$$\beta > \frac{nR_m - n_A y_{AA} - n_B(R_m - n_A)}{2n_B(R_m - n_A)} \tag{21}$$

is a sufficient condition that ensures that maximization of the utility of the agents $j \in N_B$ is characterized by $y_{BB} = R_m - n_A$, for any given $y_{AA} \in [R_m - n_B, n_A - 1]$.

Secondly, take as given an arbitrary quantity $y_{BB} \in [R_m - n_A, n_B - 1]$, and then restrict attention to the (hypothetical) problem of finding the values of y_{AA} that solve

$$\max_{y_{AA} \in [R_m - n_B, n_A - 1]} u\Big(y_{AA}, \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_A}\Big).$$

Using again Assumption 2–(4) (c), we can guarantee the solution to the problem above is characterized by $y_{AA} = R_m - n_B$ if

$$\beta > \frac{nR_m - n_A y_{AA} - n_B y_{BB}}{2n_A y_{AA}}$$

for each $y_{AA} \in [R_m - n_B, n_A - 1]$. Since the function $[nR_m - n_A y_{AA} - n_B y_{BB}]/2n_A y_{AA}$ is strictly decreasing in y_{AA}, it follows that

$$\beta > \frac{nR_m - n_A(R_m - n_B) - n_B y_{BB}}{2n_A y_{BB}} \tag{22}$$

is a sufficient condition that ensures maximization of the utility of the agents $i \in N_A$ is characterized by $y_{AA} = R_m - n_B$, for any given $y_{BB} \in [R_m - n_A, n_B - 1]$.

Crucially, if both conditions Eq. (21) and Eq. (22) are simultaneously satisfied for $y_{AA} = R_m - n_B$ and $y_{BB} = R_m - n_A$, then the utility of the agents from the smaller group N_B is maximized when they choose $y_{BB} = R_m - n_A$ conditional on the choice $y_{AA} = R_m - n_B$, while at the same time, the utility of the agents from the smaller group N_A is maximized when they choose $y_{AA} = R_m - n_B$ conditional on the choice $y_{BB} = R_m - n_A$. Since such sufficient conditions combined guarantee the above mentioned (common) features for the optimal choices of the two (hypothetical) problems relative to each of the populations, it follows that such conditions are sufficient to ensure that the only solution to the problem in Eq. (18) entails $y_{AA} = R_m - n_B$ and $y_{BB} = R_m - n_A$.

Note that Eq. (21) and Eq. (22) are simultaneously satisfied for $y_{AA} = R_m - n_B$ and $y_{BB} = R_m - n_A$ if and only if

$$\beta > \max\left\{\frac{n_A}{R_m - n_A}, \frac{n_B}{R_m - n_B}\right\} = \frac{n_A}{R_m - n_A},$$

where we are taking into account that $n_A \geq n_B$. Therefore, if

$$\beta > \frac{n_A}{R_m - n_A} = \frac{n_A}{m} = \beta_h(m),$$

then the only way in which the social planner can maximize the value function $v(g(\hat{x}))$ is by choosing

$$y_{AA} = (n_A - n_B) + m,\ y_{BB} = m,\ y_{AB} = z_{AB} = n_B,\ y_{BA} = z_{BA} = n_A.$$

Accordingly, for each agent $i \in N_\theta$, each type $\theta \in \Theta$, and $\theta' \neq \theta$, an efficient network $\hat{g} = g(\hat{x})$ entails $s_i(\hat{x}) = R_m - n_{\theta'}$ and $d_i(\hat{x}) = n_{\theta'}$. ∎

Proof of *Corollary 6*. Let $\hat{x}$ be a strategy profile that satisfies the necessary condition given by Proposition 4. Take $n_A = n_B = n/2$. Then, the problem that faces the social planner stated in Eq. (18) can be rewritten as

$$\begin{aligned} &\max_{\{y_{AA}, y_{BB}\}} V(y_{AA}, y_{BB}) \\ &\text{s.t.:}\ \ y_{AA} \in [R_m - n/2, n/2 - 1]; \\ &\qquad\ \ y_{BB} \in [R_m - n/2, n/2 - 1], \end{aligned} \tag{23}$$

where

$$V(y_{AA}, y_{BB}) \equiv u\big(y_{AA}, R_m - (1/2)(y_{AA} + y_{BB})\big) + u\big(y_{BB}, R_m - (1/2)(y_{AA} + y_{BB})\big).$$

Using the problem in Eq. (23), we proceed then as follows.

(i) Note that, for each type $\theta \in \Theta$, we have that $\partial V(y_{AA}, y_{BB})/\partial y_\theta > 0$ if and only if

$$\frac{\partial u\big(y_{\theta\theta}, R_m - (1/2)(y_{AA} + y_{BB})\big)}{\partial s_i} > \frac{\partial u\big(y_{\theta\theta}, R_m - (1/2)(y_{AA} + y_{BB})\big)}{\partial d_i}$$

Assumption 2–(4) (b) allows us to establish that the inequality above is satisfied if and only if

$$\beta < \frac{R_m - (1/2)(y_{AA} + y_{BB})}{y_{\theta\theta}}$$

or each $y_{AA}, y_{BB} \in [R_m - n/2, n/2 - 1]$. Then, for the symmetric choice $y_{AA} = y_{BB} = n/2 - 1$—in which each agent from each population group invests with full intensity in all other same-type fellows—to be associated to an efficient network, the required necessary and sufficient condition on the level of assortative interests takes the form

$$\beta < \frac{2R_m}{n-2} - 1 = \frac{2(m+1)}{n-2} = \beta_l(m),$$

as stated.

(ii) For each type $\theta \in \Theta$, we have that $\partial V(y_{AA}, y_{BB})/\partial y_\theta < 0$ if and only if

$$\frac{\partial u\big(y_{\theta\theta}, R_m - (1/2)(y_{AA} + y_{BB})\big)}{\partial s_i} < \frac{\partial u\big(y_{\theta\theta}, R_m - (1/2)(y_{AA} + y_{BB})\big)}{\partial d_i}$$

It follows from Assumption 2–(4) (c) that the inequality above is satisfied if and only if

$$\beta > \frac{R_m - (1/2)(y_{AA} + y_{BB})}{y_{\theta\theta}}$$

for each $y_{AA}, y_{BB} \in [R_m - n/2, n/2 - 1]$. Then, for the symmetric choice $y_{AA} = y_{BB} = R_m - n/2$—in which each agent from each population group invests with full intensity in all different-type agents—to be associated to an efficient network, the required necessary and sufficient condition on the level of assortative interests takes the form

$$\beta > \frac{n}{2R_m - n} = \frac{n}{2m} = \beta_h(m),$$

as stated.

(iii) Consider a level of assortative interests $\beta \in (\beta_l(m), \beta_h(m))$. Then, it follows from (i) and (ii) above that neither choices in which all agents invest with full intensity in all their same-type fellows nor choices in which they invest with full intensity in all different-type agents induce efficient networks. Now, consider symmetric aggregate investment choices $y_{AA} = y_{BB} = \hat{y}$ that give rise to partially homophilic networks that belong to the class in which all agents behave unilaterally as in [b] of Lemma 1 ([b] in Fig. 1). Such choices induce an efficient network if and only if

$$\frac{\partial u\big(y_{\theta\theta}, R_m - (1/2)(y_{AA} + y_{BB})\big)}{\partial s_i} = \frac{\partial u\big(y_{\theta\theta}, R_m - (1/2)(y_{AA} + y_{BB})\big)}{\partial d_i}$$

It follows from Assumption 2–(4) (a) that the requirement above is satisfied if and only if

$$\beta = \frac{R_m - \hat{y}}{\hat{y}} = \frac{n + 2m}{2\hat{y}} - 1 \quad \Leftrightarrow \quad \hat{y} = y_{AA} = y_{BB} = \frac{R_m}{1+\beta} = \frac{n+2m}{2(1+\beta)}$$

for $\hat{y} \in (R_m - n/2, n/2 - 1)$. Finally, note that symmetric aggregate investment choices $y_{AA} = y_{BB} = \hat{y}$ are required to ensure that the condition above holds for both population groups. ∎