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On Efficient Information Aggregation Networks

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Abstract

This paper considers a population of agents that are connected through a network that allows them to aggregate locally their pieces of private information about some uncertain (exogenous) parameter of interest. The agents wish to match their actions to the true value of the parameter and to the actions of the other agents. I ask how the design of (interim) efficient (minimally connected) networks depends on the level of complementarity in the agents' actions. When the level of complementarity is either low or high, efficient networks are characterized by a high number of different neighborhoods and, as a consequence, by low levels of connectivity. For intermediate levels of complementarity in actions, efficient networks tend to feature low numbers of highly connected neighborhoods. The implications of this paper are relevant in security environments where agents are naturally interpreted as analysts who try to forecast the value of a parameter that describes a potential threat to security.

Keywords: Networks, information aggregation, beauty-contests, strategic complementarity, efficiency
JEL Classification: C72, D83, D84, D85

Resumen

Este artículo considera una población de agentes que están conectados mediante una red a través de la cual agregan localmente su información privada sobre un parámetro (exógeno) incierto. Los agentes desean tomar acciones adecuadas al parámetro y a las acciones del resto de agentes. Este trabajo estudia cómo el diseño de redes eficientes (mínimamente conectadas) depende del nivel de complementariedad en las acciones de los agentes. Cuando el nivel de complementariedad es o bien muy bajo o bien muy alto, las redes eficientes se caracterizan por un gran número de vecindades y, en consecuencia, por bajos niveles de conectividad. Para niveles intermedios de complementariedad en acciones, las redes eficientes están compuestas por pocas vecindades, que están altamente conectadas. Las implicaciones de este artículo son relevantes en contextos de seguridad donde los agentes pueden ser interpretados como analistas que tratan de anticipar el valor de una variable que describe una amenaza de seguridad.

Palabras clave: *Redes, agregación de información, complementariedad estratégica*
JEL Classification: C72, D83, D84, D85

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Abstract

This paper considers a population of agents that are connected through a network that allows them to aggregate locally their pieces of private information about some uncertain (exogenous) parameter of interest. The agents wish to match their actions to the true value of the parameter and to the actions of the other agents. I ask how the design of (interim) efficient (minimally connected) networks depends on the level of complementarity in the agents' actions. When the level of complementarity is either low or high, efficient networks are characterized by a high number of different neighborhoods and, as a consequence, by low levels of connectivity. For intermediate levels of complementarity in actions, efficient networks tend to feature low numbers of highly connected neighborhoods. The implications of this paper are relevant in security environments where agents are naturally interpreted as analysts who try to forecast the value of a parameter that describes a potential threat to security.

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1 Introduction

In many environments of social, economic, or political interest, decision-makers seek to match their actions both to some unknown underlying variable and to the actions chosen by others. While the first motive is typically regarded as a "fundamental motive," the second motive is purely a "coordination motive."[1] The canonical framework that captures

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[1]For example, suppose that the profitability of some investment activity depends on an uncertain exogenous state of the world and on the aggregate investment. Here investors would like to pick investment strategies that match both the exogenous variable and the other investors' strategies as well.

both types of motives in strategic scenarios is that of "beauty contest" games.[2] Beauty contests are particularly suitable to capture environments where security issues are a main concern. A typical example is that of a group of analysts who independently try to forecast an exogenous variable that describes a potential threat to security. Each of the analysts wishes to follow an strategy appropriate for the true value of the uncertain variable. Since coordination helps prevent (or at least mitigate) security threats, the analyst also wishes to follow a course of action similar to the other analysts' actions.[3] In these beauty contests scenarios, decision-makers wish to collect information that helps them resolve their uncertainty about the underlying variable and the likely actions of the others.

In practice, the presence of networks is ubiquitous in contexts where agents collect and share information. When decision-makers can only interact locally through a network, the architecture of the network places restrictions on the ways in which they can aggregate their pieces of private information. The typical approach considers that decision-makers can only share information with their neighbors in the network. Often, security analysts are involved in networks where neighborhoods are teams of forecasters. In many real-world situations, a network of security analysts encompasses formal connections between different organizations as well as more informal connections based on friendship, family, or informal online relationships. In these environments, efficiency insights provide particularly useful recommendations for the design of teams of analysts, as well as for the appropriateness of establishing collaboration links between different security organizations.

This paper considers a (relatively large) population of decision-makers (or security analysts) that have access to some private signals about the underlying state and that, in addition, can share locally the information that they obtain from their signals according to their connections in a network. For this framework, I explore how the "coordination motive" influences the design of efficient networks the maximize the social welfare (or, equivalently, that minimize the social loss) of the population of agents. The efficiency benchmark used in this paper considers interim utilities. In other words, in order to design an efficient network, the central planner is able to access the information available to the agents upon receiving their signals. This seems a reasonable approach in many environments. For security contexts, it is certainly appealing when one considers that the central planner is some central institution that coordinates the teams of analysts that

[2]The "beauty contest" terminology comes originally from a well-know parable by Keynes (1936) (Chapter 12). Following the seminal contribution of Morris & Shin (2002), "beauty contest" games have been extensively used to explore a wide range of phenomena in a number of settings, including investment games (Angeletos & Pavan (2004) and Angeletos & Pavan (2007)), financial markets (Allen et al. (2006)), monopolistic competition (Hellwig & Veldkamp (2009)), or models of political leadership (Dewan & Myatt (2008)), among others.

[3]For example, under a terrorist attack threat, the analyst wishes to assess which is the most likely location of the attack but also wants to come up with locations not very distant from those predicted by other analysts. In this way, counterterrorism measures could be more effective to prevent the attack.

constitute the network.

To investigate this normative question in the context of networks, one must understand the forces behind two different mechanisms. First, how does the informative content of a fixed set of signals varies with the sizes of the neighborhoods that receive and aggregate the information contained in such signals? Secondly, how does social welfare depend on the informational content derived from the aggregation of a fixed set of signals? The first question refers to the scale properties of information aggregation within a set of agents. To answer the second question, one needs to investigate the *social value of information* when both a fundamental motive and a coordination motive are present.[4]

This paper shows in Proposition 1 that the informational content of a fixed set of signals increases monotonically as the neighborhoods that aggregate them are split into smaller neighborhoods. In other words, there are "diminishing returns" that affect the final informational content from information aggregation as neighborhoods increase in size. Interestingly enough, by selecting a network, one affects simultaneously two different aspects of information that influence the social value of information. First, as acknowledged above, the informational content of the set of available signals is affected. Secondly, by selecting the size and number of neighborhoods, one also determines whether the information finally available to the agents has either a more private or more public nature. Proposition 2 shows that, when the levels of strategic complementarity in actions are either relatively low or high, social welfare decreases with the informational content of the entire set of signals. For intermediate levels of strategic complementarity in actions, social welfare increases with the informational content of the signals of all the agents. By combining both insights, it follows that, when the coordination motive is either low or high, efficient networks are characterized by larger numbers of different neighborhoods that, therefore, give rise to relatively low connected networks. For intermediate degrees of the coordination motive, efficiency is characterized by lower numbers of distinct neighborhoods, so that high rates of connectivity are common features in efficient networks.

The rest of this paper is structured as follows. Section 2 lays out the model and Section 3 specifies the efficiency benchmark. The main results are obtained in Section 4 and Section 5 concludes. While the derivation of equilibrium is a crucial step to investigate efficiency in the proposed benchmark, it is also constructive so that the required technical details are included in the main text. Other technical details, such as the proofs of Lemma 1, Corollary 1, and of Propositions 1 and 2, are relegated to the Appendix.

[4] Angeletos & Pavan (2007) have investigated in a very comprehensive way the social value of information in an ex-ante efficiency benchmark without restrictions in the form of local interactions and where the agents have both private and public sources of information. For that environment, they have shown that whether more informational content increases or decreases welfare depends on whether equilibrium is efficient under both complete and incomplete information or only under incomplete information. Their contribution highlights that understanding the social value of information depends crucially on the notion of efficiency used. Without a well-specified efficiency benchmark, assessing the social value of information follows the folk theorem that "everything goes" in a second-best world. Assessing the social value of information with complementarities in the presence of networks remains a question far from understood.

2 Model

There is a (measure–1) continuum of agents, indexed by $i \in [0,1]$, that wish to estimate an unknown *state of the world* $\theta \in \mathbb{R}$. Each agent receives a utility according to a common (quadratic) *utility function* $u(a,\theta)$ that depends on an *action profile* $a : [0,1] \to \mathbb{R}$, where $a(i)$ is the action chosen by agent i, and on the state θ. In security environments, we can think of agents as security analysts or forecasters.

2.1 Preferences and Information Structure

The main intuition is conveyed using a beauty contest game (as in Morris & Shin (2002), Angeletos & Pavan (2007), Hellwig & Veldkamp (2009), Dewan & Myatt (2008), and Jimenez-Martinez (2014)). Each agent i's utility is given by

$$u(a,\theta) = -\left(a(i) - (1-\lambda)\theta - \lambda \int_0^1 a(h)dh\right)^2, \tag{1}$$

where $\lambda \in (0,1)$ is a parameter that measures the degree of strategic complementarity in the agents' actions. Intuitively, λ captures the relative importance of the coordination motive in the agent's utility. Higher values of λ indicate higher levels of strategic complementarity.

The state of the world θ is unknown to the agents and, for tractability reasons, the underlying information structure is assumed to be Gaussian with $\theta \sim \mathcal{N}(0,\sigma^2)$. Agents receive some (exogenous) information about θ from a finite set of *noisy signals* $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ throughout two periods $t \in \{0,1\}$.[5] Specifically, signals are (exogenously) assigned to agents according to a finite partition $\{N_1, N_2, \ldots, N_n\}$ of the set of agents $[0,1]$, where $N_1 = [0, i_1]$, $N_j = (i_{j-1}, i_j]$ for each $j \in \{2,\ldots,n-1\}$, and $N_n = (i_{n-1}, 1]$. Then, at $t=0$ each agent $i \in N_j$ receives a noisy signal $s_j = \theta + \varepsilon_j$, where $\varepsilon_j \sim \mathcal{N}(0, \pi_j^{-1})$. Thus, π_j measures the precision of the signal received by any agent $i \in N_j$. Each noise term ε_j is independent of the state θ and the noise terms $\{\varepsilon_j\}_{j=1}^n$ are independent from each other as well. Signals are assumed to be (conditional on the state) independent (i.e., the conditional random variables $\{(s_j \,|\, \theta)\}_{j=1}^n$ are independent).[6]

2.2 The Network Structure

After receiving their initial information, agents can communicate at $t=1$ according to a *network*. The network is a directed graph that specifies the sets of agents that each agent can listen to. Specifically, a network is described by means of a *network function*

[5] Although the set of agents is a continuum, I assume that agents have access to a finite number of signals for technical tractability.

[6] Of course, signals cannot be unconditional independent because all of them depend on the state of the world.

$g : [0,1] \to \Delta([0,1])$ where $g(i)$ indicates *agent i's neighborhood* or the set of agents that i can listen to. In security environments, we can think of a neighborhood as a team of security analysts or forecasters or, alternatively, as a group of different security organizations that collaborate in some security project. Each agent listens to himself, $i \in g(i)$. Let $\Upsilon_g = g([0,1])$ denote the image set of the network function g, or the set of possible neighborhoods in the network, and let us use υ, τ, and δ to denote generic neighborhoods throughout the paper. To avoid trivial cases, I will restrict attention in this paper to networks that are *minimally connected* in the sense that $\upsilon \cap \tau \neq \emptyset$ for each $\upsilon, \tau \in \Upsilon_g$.

For a network function g, all agents in a given neighborhood $\upsilon \in \Upsilon_g$ can observe at $t = 1$ the same signals.[7] After appropriately relabeling the signals from the set $\mathcal{S}$ of available signals, let $s(\upsilon)$ denote a *restricted signal profile*, which simply consists of the string of signals observed by all agents in neighborhood υ. Since the set of available signals $\mathcal{S}$ is finite, each restricted signal profile $s(\upsilon)$ must be finite as well.

2.3 Optimal Actions

Given the informational constraints imposed by the network function g, the agents are engaged in a game where each agent $i \in [0,1]$ that belongs to some neighborhood $\upsilon \in \Upsilon_g$ chooses at $t = 1$ an action $a^*(i)$ so as to maximize his conditional expected utility $E\big[u(a,\theta) \,\big|\, s(\upsilon)\big]$. Under the preference specification in (1), a *Bayesian Nash equilibrium* (BNE) is a function $a^* : [0,1] \to \mathbb{R}$ such that, for each neighborhood $\upsilon \in \Upsilon_g$, each agent $i \in \upsilon$ solves the problem

$$\min_{a(i)\in\mathbb{R}} \quad E\left[\left(a(i) - (1-\lambda)\theta - \lambda \int_{\tau\in\Upsilon_g} \int_{h\in\tau} a(h)\, dh\, d\tau\right)^2 \,\Bigg|\, s(\upsilon)\right].$$

Nonetheless, we can restrict attention to symmetric BNE where all agents that belong to some common neighborhood (and that, therefore, observe the same signals) optimally choose the same action. To see this, suppose that some agents in a given neighborhood τ choose different optimal actions. Then, the expectation that the agents of some other neighborhood υ have about the average optimal action followed in neighborhood τ is

$$E\left[\int_{h\in\tau} a^*(h)\, dh \,\bigg|\, s(\upsilon)\right] = \int_{h\in\tau} E\left[a^*(h) \,|\, s(\upsilon)\right]\, dh$$

This expectation depends on the restricted signal profile $s(\upsilon)$ but not on the names of the agents $i \in \upsilon$. All agents that belong to some common neighborhood υ aggregate the same information and obtain some common posteriors on θ and on the actions chosen by

[7] Thus, I am assuming that there is no loss of information in the communication process that takes place within each neighborhood.

other agents. In addition, since the loss function that the agents in υ are minimizing is strictly convex, the corresponding best-reply must be unique. Therefore, we must have

$$\int_{h\in\tau} E\left[a^*(h)\,|\,s(\upsilon)\right]\,dh = E\big[a^*(\tau)\,\big|\,s(\upsilon)\big],$$

where $a^*(\tau)$ indicates the optimal action chosen by any agent that belongs to neighborhood τ. Notice that the optimal action $a^*(\tau)$ depends only on the subset of signals $s(\tau)$.[8] From here onwards, let us use for simplicity $E_\upsilon[\cdot]$ and $\mathrm{Var}_\upsilon[\cdot]$ to indicate, respectively, the conditional expectation $E[\cdot\,|\,s(\upsilon)]$ and the conditional variance $\mathrm{Var}[\cdot\,|\,s(\upsilon)]$ operators for all agents in a neighborhood υ. Given the previous observations, an *action function* $a^* : \Upsilon_g \to \mathbb{R}$ is a *symmetric BNE* if and only if each action $a^*(\upsilon)$ satisfies

$$a^*(\upsilon) = (1-\lambda)E_\upsilon[\theta] + \lambda\int_{\tau\in\Upsilon_g} E_\upsilon\big[a^*(\tau)\big]\,d\tau. \tag{2}$$

Given our focus on symmetric BNE, we can express, for a fixed network function g, a BNE action profile a^* as a function $a^* : \Upsilon_g \to \mathbb{R}$ where $a^*(\upsilon)$ indicates the optimal action chosen by all agents in neighborhood g. For each $\upsilon \in \Upsilon_g$, the action $a^*(\upsilon)$ satisfies equation (2).

2.4 Social Welfare

Consider that, after period $t = 0$ and before period $t = 1$, a *central planner* designs the structure of the network g. Thus, the social planner has access to the signals available to all agents in the population. Notice that the expected loss of all agents that belong to a neighborhood υ under a symmetric BNE action function a^*, is

$$E_\upsilon\left[\left(a^*(\upsilon) - (1-\lambda)\theta - \lambda\int_{\upsilon\in\Upsilon_g} a^*(\upsilon)\,d\upsilon\right)^2\right]. \tag{3}$$

The goal of this paper is to investigate how the central planner optimally chooses g so as to minimize the *social welfare loss function*

$$L(g) = \int_{\upsilon\in\Upsilon_g} E_\upsilon\left[\left(a^*(\upsilon) - (1-\lambda)\theta - \lambda\int_{\upsilon\in\Upsilon_g} a^*(\upsilon)\,d\upsilon\right)^2\right]d\upsilon. \tag{4}$$

To address this central question, we need first to characterize the class of linear symmetric BNE of the game that the agents play once they receive their signals at $t = 1$ according to the restrictions imposed by the network. To obtain a solution to equation (2), we first need to study how information is aggregated within neighborhoods and

[8] Formally, it would be more precise to use $a^*(s(\tau))$. Yet, without loss of generality, we can drop the term $s(\cdot)$ to simplify notation.

how this influences the agents' optimal actions. For a network function g and for each neighborhood $\upsilon \in \Upsilon_g$, the pairs $(\theta, s(\upsilon))$ are jointly normally distributed. Let us use $\mathrm{Cov}[\theta, s(\upsilon)]$ to denote the vector of covariances between the state of the world and each of the signals observed in neighborhood υ and $\mathrm{Var}[s(\upsilon)]$ to denote the variance-covariance matrix of the signals in $s(\upsilon)$. It follows from some basic results on normal distributions that

$$E_\upsilon[\theta] = \mathrm{Cov}[\theta, s(\upsilon)]' \cdot \mathrm{Var}[s(\upsilon)]^{-1} \cdot s(\upsilon) \tag{5}$$

and

$$\mathrm{Var}_\upsilon[\theta] = \sigma^2 - \mathrm{Cov}[\theta, s(\upsilon)]' \cdot \mathrm{Var}[s(\upsilon)]^{-1} \cdot \mathrm{Cov}[\theta, s(\upsilon)]. \tag{6}$$

Hence, normality ensures that the conditional expectations of the state are linear in the signals $s(\upsilon)$ observed in neighborhood υ. This implication allows us to focus the analysis of BNE on linear strategies. If the agents in one neighborhood τ use a linear strategy with respect to the signals that they observe $s(\tau)$, then the optimal action that the agents follow in neighborhood υ must be also linear in the signals $s(\upsilon)$. While linear strategies are fairly simple and intuitive to interpret, in the current context they are also robust.

3 The Efficient Network Design Problem

This section derives the agents' optimal actions in a symmetric BNE and the corresponding social welfare loss function. Since the technical details required to set up the problem that the social planner addresses are constructive, they are provided here in the main text. Equation (2) reveals that the optimal action followed by the agents of a given neighborhood υ depend in a recursive way on the average posterior expectation over the true state. Hence, we need to account for arbitrarily higher-order average posterior expectations over θ. To formalize these average posterior expectations, let $\bar{E}[\theta] = \int_{\upsilon \in \Upsilon_g} E_\upsilon[\theta]\, d\upsilon$ be the average posterior expectation on the state over neighbourhoods.[9] We begin with the 0–order average posterior expectation. Notice that the 0–order average posterior expectation must coincide with the true realization of the state so that we set $\bar{E}^{(0)}[\theta] = \theta$. Then, for the 1–order average posterior expectation, we have

$$\bar{E}^{(1)}[\theta] = \bar{E}\left[\bar{E}^{(0)}[\theta]\right] = \bar{E}[\theta] = \int_{\upsilon \in \Upsilon_g} E_\upsilon[\theta]\, d\upsilon,$$

whereas for higher-order average posterior expectations, we use $\bar{E}^{(m)}[\theta] = \bar{E}\left[\bar{E}^{(m-1)}[\theta]\right]$ to indicate in a recursive way the m–order average posterior expectation over θ, for $m \geq 2$. With such higher-order average posterior expectations in place, recursive application of

[9]Since this is an average over all neighborhoods, $\bar{E}[\theta]$ equivalently indicates the average posterior expectation on θ over all agents.

equation (2) allows us to express the optimal action followed in neighborhood υ as

$$\begin{aligned} a^*(\upsilon) &= (1-\lambda)\Big[E_\upsilon\big[\bar{E}^{(0)}[\theta]\big] + \lambda E_\upsilon\big[\bar{E}^{(1)}[\theta]\big] + \lambda^2 E_\upsilon\big[\bar{E}^{(2)}[\theta]\big] + \cdots\Big] \\ &= (1-\lambda)\sum_{m=0}^{\infty}\lambda^m E_\upsilon\big[\bar{E}^{(m)}[\theta]\big]. \end{aligned} \tag{7}$$

Under the assumed information structure, we have $\mathrm{Cov}[\theta, s(\upsilon)] = \sigma^2\underline{1}$, where $\underline{1}$ is a vector of ones with the same dimension as the profile of signals $s(\upsilon)$. Also, recall that $s_j = \theta + \varepsilon_j$, where $E_\upsilon[\varepsilon_j] = 0$ for all signals $j = 1, \ldots, n$. Take a given realization of the state θ. Then, using the expression in (5), we obtain that $E_\upsilon\big[\bar{E}^{(0)}[\theta]\big] = E_\upsilon[\theta]$, for the 0–order average posterior expectation, and

$$\begin{aligned} E_\upsilon\big[\bar{E}^{(1)}[\theta]\big] &= E_\upsilon\left[\left(\sigma^2\int_{\upsilon\in\Upsilon_g}\underline{1}'\cdot\mathrm{Var}[s(\upsilon)]^{-1}\cdot\underline{1}\,d\upsilon\right)\theta\right] \\ &= \left(\sigma^2\int_{\upsilon\in\Upsilon_g}\underline{1}'\cdot\mathrm{Var}[s(\upsilon)]^{-1}\cdot\underline{1}\,d\upsilon\right)E_\upsilon[\theta], \end{aligned}$$

for the 1–order average posterior expectation. Here again, $\underline{1}$ is a vector of ones with the same dimension as the profile of signals $s(\upsilon)$. Let us use

$$\overline{\omega}_g = \sigma^2\int_{\upsilon\in\Upsilon_g}\underline{1}'\cdot\mathrm{Var}[s(\upsilon)]^{-1}\cdot\underline{1}\,d\upsilon$$

to denote the average of the inverses of the posterior variances of the state across neighborhoods in the network. Given this notation for the average across (the inverse of) posterior variances, we can write $\bar{E}[\theta] = \overline{\omega}_g\theta$ and iterate to obtain that $\bar{E}^{(m)}[\theta] = \overline{\omega}_g^m\theta$ for each $m \geq 0$. Thus, we can express the equality in (7) as

$$\begin{aligned} a^*(\upsilon) &= (1-\lambda)\left[1 + \lambda\overline{\omega}_g + \lambda^2\overline{\omega}_g^2 + \cdots\right]\,E_\upsilon[\theta] \\ &= \left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g}\right)E_\upsilon[\theta], \end{aligned} \tag{8}$$

where $E_\upsilon[\theta]$ satisfies the equality in (5). Now, if we average the expression above over all neighborhoods in the network, we obtain

$$\begin{aligned} \int_{\upsilon\in\Upsilon_g} a^*(\upsilon)\,d\upsilon &= \left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g}\right)\int_{\upsilon\in\Upsilon_g}E_\upsilon[\theta]\,d\upsilon \\ &= \left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g}\right)\overline{\omega}_g\,\theta. \end{aligned}$$

Therefore, in a BNE, each agent that belongs to a neighborhood υ wishes to match his action to the objective

$$(1-\lambda)\theta + \lambda\int_{\upsilon\in\Upsilon_g}a^*(\upsilon)\,d\upsilon = \left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g}\right)\theta \tag{9}$$

By plugging the expressions in (8) and (9) into the expected loss function given by (3), we obtain:

$$E_\upsilon\left[\left(\frac{(1-\lambda)E_\upsilon[\theta]}{1-\lambda\overline{\omega}_g}-\frac{(1-\lambda)\theta}{1-\lambda\overline{\omega}_g}\right)^2\right]=\left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g}\right)^2\mathrm{Var}_\upsilon[\theta],$$

where the conditional variance $\mathrm{Var}_\upsilon[\theta]$ is given by the expression in (6). By combining this with the expression in (4), we obtain the the social welfare loss function is given by

$$L(g)=\left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g}\right)^2\int_{\upsilon\in\Upsilon_g}\mathrm{Var}_\upsilon[\theta]\,d\upsilon.$$

Before proceeding further with the analysis, it is convenient to comment on one important modeling requirement. In order to explore the role of the network on social welfare, we need to consider that the set of possible networks Υ_g is finite. Intuitively, each neighborhood υ contributes to the social welfare value by aggregating the information contained in its members' private signals. Then, the precision of the information aggregated in this way is summarized by the posterior variance $\mathrm{Var}_\upsilon[\theta]$. With an infinite set of possible neighborhoods, the posterior variances of the different neighborhoods can be aptly aggregated using an average of posterior variances. But averaging over the neighborhoods' posterior variances vanishes away the informative contribution of each neighborhood to the social welfare value. In this sense, our research question becomes irrelevant with an infinite number of possible networks.[10] This is very intuitive: exploring the value (in terms of information) that each neighborhood contributes to social welfare is not appealing with an infinite number of neighborhoods. Since we are interested in exploring how the network structure affects social welfare, the analysis will restrict attention to finite sets Υ_g of possible neighborhoods.

Another technical comment, related to the previous one, is also in order here. Recall that our derivation of symmetric BNE has made use of the law of large numbers to average expectations on the state over neighborhoods. Therefore, we must consider that the number of neighborhoods, though finite, is sufficiently large for our formal arguments to be appealing in the environment that we are studying. The analysis of higher-order beliefs used in this paper builds on the approach followed, among others, by Morris & Shin (2002), Angeletos & Pavan (2007), Hellwig & Veldkamp (2009), and Dewan & Myatt (2008). As in these papers, the formal analysis used here also invokes the law of large numbers.[11]

[10]Formally, if we have an infinite number of possible networks, then using our notation $\overline{\omega}_g$ for the average of the (inverses) of posterior variances across neighborhoods, it can be verified from expression (6) that $\int_{\upsilon\in\Upsilon_g}\mathrm{Var}_\upsilon[\theta]\,d\upsilon=\sigma^2[1-\overline{\omega}_g]$. In this case, the social welfare loss function can be expressed as $L(g)=\sigma^2(1-\lambda)^2(1-\overline{\omega}_g)(1-\lambda\overline{\omega}_g)^{-2}$, where $\overline{\omega}_g$ does not depend on the network structure. As a consequence, the network structure plays no role in determining the value of the social welfare.

[11]For applications where one considers instead a relatively small number of neighborhoods, the law of large numbers cannot be reasonably invoked to compute averages of expectations on the state. In

With the considerations above in place, under the restriction that the number of neighborhoods in the network is relatively large but finite, the social welfare loss function is given, as a function only of the model's primitives, by the expression[12]

$$L(g) = \left(\frac{1-\lambda}{1-\lambda\overline{\omega}_g} \right)^2 \sum_{\upsilon \in \Upsilon_g} \mathrm{Var}_\upsilon[\theta], \tag{10}$$

where

$$\overline{\omega}_g = \sigma^2 \sum_{\upsilon \in \Upsilon_g} \underline{1}' \cdot \mathrm{Var}[s(\upsilon)]^{-1} \cdot \underline{1}. \tag{11}$$

The objective of the social planner is to choose the network function g so as to minimize the social loss given by (10) above. Let us use $\gamma_g = |\Upsilon_g|$ to indicate the number of neighborhoods in the social network described by g.

4 Main Results

Let us use $s(\upsilon) = (s_1(\upsilon), \ldots, s_{k_\upsilon}(\upsilon))$ to account for the different signals that are included in a restricted signal profile $s(\upsilon)$ observed in a neighborhood υ. Thus, we are using k_υ to indicate the number of signals available in neighborhood υ. Also, in accordance with the earlier notation, I will use $\pi_j(\upsilon)$ to indicate the inverse of the variance of the noise term associated to $s_j(\upsilon)$, for each $j = 1, \ldots, k_\upsilon$ and each neighborhood $\upsilon \in \Upsilon_g$. Thus, $\pi_j(\upsilon)$ indicates the precision of the corresponding signal in neighborhood υ.

The informative content of the signals available to a neighborhood υ can be conveniently described by the number

$$w(\upsilon) := \sigma^2 \underline{1}' \cdot \mathrm{Var}[s(\upsilon)]^{-1} \cdot \underline{1},$$

which identifies sum of all the entries of the inverse of the variance-covariance matrix of the signal profile $s(\upsilon)$ (weighted using the variance of the state of the world). Then, notice

these cases, keeping track of the higher-order beliefs that are required to characterize equilibria follows a completely different approach. In particular, under certain conditions, one can make use of the iterated application of a knowledge index matrix. The idea of using a knowledge index matrix to track individual arbitrarily higher-order beliefs depending on the position of a relatively small number of agents in a network was originally proposed by Calvó-Armengol & de Martí-Beltran (2009). An application of the knowledge index matrix to information acquisition problems in small populations has been recently provided by Jimenez-Martinez (2014).

[12]Formally, when the number of neighborhoods is finite, the social welfare expression obtained in (10) is always an approximation to the actual one. In applications, though, the law of large numbers can be reasonably invoked (and thus the derived implications are robust) when the number of neighborhoods is large. The approximation that we are considering is more precise as the number of neighborhoods increase (and, in the limit, the law of large of numbers gives us an exact account when the number of neighborhoods tends to infinite). However, as argued earlier, considering an infinite number of neighborhoods would make our research question irrelevant.

that we can express the average of the inverses of the posterior variances of the state in the network as $\overline{\omega}_g = \sum_{\upsilon\in\Upsilon_g} w(\upsilon)$. Intuitively, each $w(\upsilon)$ is simply a scalar that gives us some information about the joint precision of the signal profile $s(\upsilon)$. Higher values of $w(\upsilon)$ are associated to lower degrees of noise in the corresponding signals and, therefore, to more informative signal profiles. The following lemma provides formally this implication.

Lemma 1. *The scalar $w(\upsilon) = \sigma^2\underline{1}' \cdot \mathrm{Var}[s(\upsilon)]^{-1} \cdot \underline{1}$ lies in the interval $(0, 1/\sigma^2)$ and it increases (strictly) with the sum of the precision of all the signals observed in neighborhood υ, $\sum_{j=1}^{k_\upsilon} \pi_j(\upsilon)$.*

To obtain our main insights about efficient information aggregation networks, I turn now to derive a more tractable expression for the social welfare loss. From the expression in (6), and under our normality assumptions, it follows that the posterior average of posterior variances of the state across neighborhoods is given by

$$\sum_{\upsilon\in\Upsilon_g} \mathrm{Var}_\upsilon[\theta] = \sigma^2\Big(\gamma_g - \sum_{\upsilon\in\Upsilon_g} w(\upsilon)\Big) = \sigma^2\left(\gamma_g - \overline{\omega}_g\right).$$

Therefore, using the expression in (10) for the social welfare loss, the problem that the social planner faces in order to design efficient networks can be neatly expressed as

$$\min_g \; L(g) := \sigma^2(1-\lambda)^2 \frac{(\gamma_g - \overline{\omega}_g)}{(1-\lambda\overline{\omega}_g)^2}, \qquad \text{where } \overline{\omega}_g = \sum_{\upsilon\in\Upsilon_g} w(\upsilon). \tag{12}$$

The problem in (12) above establishes the crucial incentives for the social planner to design (interim) efficient networks. Before presenting the main results, let me briefly make two comments on such a design problem.

First, notice that, by choosing the network function g, the planner chooses the number of neighborhoods γ_g in the network. Yet, recall that we are ruling out from the analysis the complete network (and, in general, very highly connected networks that yield small numbers of neighborhoods) in order to consider a sufficiently large number of neighborhoods. Furthermore, since $\sum_{\upsilon\in\Upsilon_g} \mathrm{Var}_\upsilon[\theta] = \sigma^2\left(\gamma_g - \overline{\omega}_g\right) > 0$, we must have also $\gamma_g > \overline{\omega}_g$ as an internal condition of the social planner's problem. Then, one approach that the social planner can take solve the problem in (12) is to begin by choosing the average of inverse variances $\overline{\omega}^*_g$ so as to minimize $L(g)$ for a constant value of γ_g. After that, the social planner can decide on the best way to distribute the network neighborhoods in order to achieve the desired $\overline{\omega}^*_g = \sum_{\upsilon\in\Upsilon^*_g} w^*(\upsilon)$. Following this approach, the next proposition provides a key result to explore how the social planner designs the network after setting the desired goal for the average of inverse variances $\overline{\omega}^*_g$.

Proposition 1. *The informational content of the collection of signals received in any neighborhood $\upsilon \in \Upsilon_g$ is strictly smaller than the aggregation of informational contents obtained by splitting υ into any two disjoint neighborhoods, that is, $w(\upsilon) < w(\tau) + w(\delta)$ for any $\tau, \delta \in \Upsilon_g$ such that $\upsilon = \tau \cup \delta$ with $\tau \cap \delta = \emptyset$.*

The informational content of a set of signals naturally increases when the number of signals in the set increases, as can be derived directly from Lemma 1. However, in principle it is unclear how the informational content of a *fixed set of signals* changes when the neighborhoods through which the information is aggregated vary. Proposition 1 above offers a type of "diminishing returns" result under which the final informational content increases monotonically as we split neighborhoods into smaller neighborhoods. Larger neighborhoods seem to feature a type of "congestion" to aggregate information. The message conveyed here for security environments is that we want to split teams of security analysts into smaller teams if we are interested in increasing the overall informational of their private sources of information.

The result of Proposition 1 is very interesting to consider the strategies that one would like to follow in order to design efficient information aggregation networks. Far from obvious, it seems somehow surprising as it establishes that the overall informational content described by the term $\overline{\omega}_g$ always increases by separating neighborhoods into smaller neighborhoods. Each of the resulting smaller neighborhoods receive accordingly a smaller number of signals. The following corollary follows by combining some of the insights provided by Lemma 1 and Proposition 1 above.

Corollary 1. *For any network function g, the average of the inverses of the posterior variances of the state $\overline{\omega}_g$ lies in the interval $(\underline{\omega}, \gamma_g \min\{1, 1/\sigma^2\})$, where the lower bound $\underline{\omega}$ is specified as:*

$$\underline{\omega} := \frac{\sum_{j=1}^{n} \pi_j}{1 + \sigma^2 \sum_{j=1}^{n} \pi_j}.$$

The final insights of this paper on efficient information aggregation networks are provided by the following proposition. Since this paper focuses on networks with a relatively high number of neighborhoods, the proposition now states explicitly this requirement. From the expression for the social loss stated in (12) notice that more aggregate informative content (i.e., higher values of the variable $\overline{\omega}_g$) raises welfare by lowering the term $(\gamma_g - \overline{\omega}_g)$ but it harms welfare by raising the term $1/(1 - \lambda\overline{\omega}_g)^2$. In principle, it is unclear which of the two effects dominates and the answer clearly depends on the level of complementarity captured by $\lambda \in (0, 1)$.

Proposition 2. *Suppose that the number of neighborhoods in the network is no less than some sufficiently large bound, $\gamma_g \geq \underline{\gamma}$. Then,*

(i) for levels of strategic complementary either relatively low, $\lambda \in \left(0, \frac{1}{2n - \overline{\omega}}\right)$, or relatively high, $\lambda \in \left(\frac{1}{\overline{\omega}}, 1\right)$, the optimal network is the connected line where the $\overline{\omega}$ is the information level that maximizes $\overline{\omega}_g = \sum_{\upsilon \in \Upsilon_g} \underline{1}' \cdot \mathrm{Var}[s(\upsilon)]^{-1} \cdot \underline{1}$, whereas

(ii) for intermediate levels of strategic complementarity $\lambda \in \left(\frac{1}{2\underline{\gamma} - \underline{\omega}}, \frac{1}{\underline{\omega}}\right)$ the optimal network is the highest connected possible network under the requirement that the number of

resulting neighborhoods is $\underline{\gamma}$.

Proposition 2 gives us sufficient conditions under which interesting "corner solutions" to the planner's problem described in (12) arise. In particular, the levels of complementarity stated in (i) and (ii) of Proposition 2 guarantee that the social welfare loss function $L(g)$ is (strictly), respectively, increasing and decreasing in the variable $\overline{\omega}_g$ that summarizes the informational content of all signals available to the agents.

As shown by Angeletos & Pavan (2007), the mechanisms behind the social value of information are complex and they depend on two basic considerations. The insights that one can draw about efficiency in the presence of public and private information rely very much on the proposed efficiency benchmark. For an efficiency benchmark that uses ex-ante utility, whether more information is always welfare-improving depends crucially on whether equilibrium is efficient under both complete and incomplete information or only under incomplete information. When equilibrium is efficient under both complete and incomplete information, more information (either public or private) is always welfare-improving.[13] Our efficiency benchmark is quite different since it uses interim utilities instead. Yet, in our setting, the presence of the network also introduces a dimension where information can, somehow, be considered as (endogenously) "more or less public." If the number of neighborhoods is relatively small (so that the resulting network is highly connected), then information becomes relatively more public as more agents have access to a common set of signals. In this case, as shown by Proposition 1, the informational content obtained from the aggregation of all signals is relatively low. On the other hand, if the number of neighborhoods is large (so that the resulting network is poorly connected), then the information content obtained from the aggregation of signals is relatively high but the nature of the information available to the agents is mainly private. For instance, in the case of the connected line network, agents have access only to two (on the extremes of the line) or to three (in the neighborhoods not located on the extremes of the line) signals.

Proposition 2 above establishes that when the agents wish to follow a course of action very close either to the fundamental parameter or to the actions followed by other agents, the social value of information increases with the informational content obtained from the aggregation of all private signals available in the population. In short, when the agents wish to match *only one of their goals* (either the underlying state of the world or other agents' actions), improving social welfare requires higher aggregate levels of information and that information received in each neighborhood has a more private nature. Here the social planner is willing to sacrifice less volatility in the errors of the signals for higher overall accuracy. Intuitively, in this case agents are able, without consulting much with others, to forecast with relatively high precision the parameter. Recall that even when

[13] As argued by Angeletos & Pavan (2007), this is the case because equilibrium coincides with the solution to the planner's problem. Then, following an argument similar to Blackwell's Theorem, any source of additional information is welfare-improving.

only aligning actions matters, a precise forecast of the parameter indirectly helps agents to forecast better other agents' actions.

On the other hand, when the agents wish to match *simultaneously both of their goals*, social welfare increases as the informational content that results from the aggregation of all private signals available in the population lowers and, at the same time, the nature of the information available in each neighborhood becomes more public. The intuition in this case is that the population, as a whole entity (represented by the social planner), prefers to have less but more public information so that, in average, agents meet both of their incentives. In other words, the social planner prefers less volatility in the errors of the signals errors at the expense of lower overall accuracy.

5 Concluding Comments

This paper has explored the design of (interim) efficient networks in environments where decision-makers are connected through a network and where all of them (commonly) feature both a fundamental motive and a coordination motive in their preferences. Investigating this topic is interesting when one considers networks with a finite but relatively high number of neighborhoods. Otherwise, the problem turns out to be either irrelevant (when the number of neighborhoods tend to infinity so that the contribution of each neighborhood to the social value of information vanishes) or intractable (when the number of neighborhoods is relatively small so that the law of large numbers cannot be reasonably invoked to obtain a closed expression of equilibrium actions). In addition, the research question addressed here remains interesting when one uses an interim efficiency benchmark. Using the ex-ante efficiency approach instead, one would obtain directly that the efficient network is the one with the highest possible connectivity. In our model, such a network would have a number of neighborhoods equal to the minimum bound $\underline{\gamma}$. The reason behind this implication is that, for the case of beauty contest preferences, ex-ante efficiency requires the central planner to solve the same problem that faces each agent. As a consequence, any additional source of information would be welfare-improving.

Under the conditions above, this paper has shown that, when the levels of complementarity in actions are either low or high, efficient networks are characterized by a large number of neighborhoods and by low levels of connectivity. The connected line network arises as the efficient network under the requirement that networks are minimally connected. When, the levels of complementarity in actions are intermediate, efficient networks feature a relatively small number of neighborhoods and high levels of connectivity. The implications of this paper are useful to provide efficiency recommendations for networks of security analysts that are coordinated by a central institution under the requirement that such an institution can access the pieces of private information available to the analysts.

A natural extension of the setting explored here would be that of considering substitutive actions as well. This possibility seems reasonable in certain contexts but it is perhaps not very appealing in security environments where coordination helps to prevent security threats. At a more general level, understanding the social value of information when information can be aggregated only locally within neighborhoods in networks, under different efficiency benchmarks and preference specifications, remains an interesting and rather unexplored question.

Appendix

Proof of Lemma 1. Under our normality assumptions, we have

$$\mathrm{Var}[s(\upsilon)] = \begin{pmatrix} \sigma^2 + \pi_1^{-1}(\upsilon) & \sigma^2 & \cdots & \sigma^2 \\ \sigma^2 & \sigma^2 + \pi_2^{-1}(\upsilon) & \cdots & \sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2 & \sigma^2 & \cdots & \sigma^2 + \pi_{k_\upsilon}^{-1}(\upsilon) \end{pmatrix} = \sigma^2 \underline{1} \cdot \underline{1}' + D,$$

where D is the diagonal matrix $D = \mathrm{diag}\left(\pi_j^{-1}(\upsilon)\right)_{j=1,\ldots,k_\upsilon}$ that contains the variances of the noises of the signals observed in neighborhood υ. Notice that

$$D^{-1} = \begin{pmatrix} \pi_1(\upsilon) & 0 & \cdots & 0 \\ 0 & \pi_2(\upsilon) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi_{k_\upsilon}(\upsilon) \end{pmatrix}.$$

Using a version of the Sherman–Morrison's formula to compute the inverse of a sum of matrices, (see, e.g., Sherman & Morrison (1950) and Henderson & Searle (1981)), we obtain

$$\mathrm{Var}[s(\upsilon)]^{-1} = D^{-1} - \frac{\sigma^2}{1 + \sigma^2 \underline{1}' \cdot D^{-1} \cdot \underline{1}} \left[D^{-1} \cdot \underline{1} \cdot \underline{1}' \cdot D^{-1}\right].$$

It follows that

$$\frac{\sigma^2}{1 + \sigma^2 \underline{1}' \cdot D^{-1} \cdot \underline{1}} = \frac{\sigma^2}{1 + \sigma^2 \sum_{j=1}^{k_\upsilon} \pi_j(\upsilon)}$$

and

$$\left[D^{-1} \cdot \underline{1} \cdot \underline{1}' \cdot D^{-1}\right] = \begin{pmatrix} \pi_1^2(\upsilon) & \pi_1(\upsilon)\pi_2(\upsilon) & \cdots & \pi_1(\upsilon)\pi_{k_\upsilon}(\upsilon) \\ \pi_2(\upsilon)\pi_1(\upsilon) & \pi_2^2(\upsilon) & \cdots & \pi_2(\upsilon)\pi_{k_\upsilon}(\upsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{k_\upsilon}(\upsilon)\pi_1(\upsilon) & \pi_{k_\upsilon}(\upsilon)\pi_2(\upsilon) & \cdots & \pi_{k_\upsilon}^2(\upsilon) \end{pmatrix}.$$

By doing the algebra, it then follows that

$$\mathrm{Var}[s(\upsilon)]^{-1} = \frac{1}{1 + \sigma^2 \sum_{j=1}^{k_\upsilon} \pi_j} \times$$
$$\begin{pmatrix} \pi_1 + \sigma^2 \sum_{j \neq 1} \pi_1 \pi_j & -\sigma^2 \pi_1 \pi_2 & \cdots & -\sigma^2 \pi_1 \pi_{k_\upsilon} \\ -\sigma^2 \pi_2 \pi_1 & \pi_2 + \sigma^2 \sum_{j \neq 2} \pi_1 \pi_j & \cdots & -\sigma^2 \pi_2 \pi_{k_\upsilon} \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma^2 \pi_{k_\upsilon} \pi_1 & -\sigma^2 \pi_{k_\upsilon} \pi_2 & \cdots & \pi_{k_\upsilon} + \sigma^2 \sum_{j \neq k_\upsilon} \pi_{k_\upsilon} \pi_j \end{pmatrix},$$

where the arguments (υ) have been conveniently dropped to simplify the expression. Therefore, in order to obtain a closed expression for the scalar $w(\upsilon) = \underline{1}' \cdot \mathrm{Var}[s(\upsilon)]^{-1}\underline{1}$, we need to aggregate all the entries of the matrix obtain above. We obtain

$$w(\upsilon) = \frac{\sum_{j=1}^{k_\upsilon} \pi_j(\upsilon)}{1 + \sigma^2 \sum_{j=1}^{k_\upsilon} \pi_j(\upsilon)} \in (0, \sigma^{-2}). \tag{13}$$

We observe directly that higher values of the (additive) aggregation of the precision of the signals $\sum_{j=1}^{k_\upsilon} \pi_j(\upsilon)$ received in neighborhood υ determine higher values of the scalar $w(\upsilon)$. ∎

Proof of Proposition 1. Let us use $\pi_\upsilon = \sum_{j=1}^{k_\upsilon} \pi_j(\upsilon)$ as a shorthand notation for the sum of the precision of the signals observed in a neighborhood $\upsilon \in \Upsilon_g$. Consider a neighborhood υ and suppose that we split it into two disjoint neighborhoods τ, δ. Thus, we are considering $\upsilon = \tau \cup \delta$ with $\tau \cap \delta = \emptyset$. First, notice that $\pi_\upsilon = \pi_\tau + \pi_\delta$. Secondly, it follows from the expression (13) obtained in the proof of Lemma 1 that

$$\begin{aligned} w(\tau) + w(\delta) - w(\upsilon) &= \frac{\pi_\tau}{1 + \sigma^2 \pi_\tau} + \frac{\pi_\delta}{1 + \sigma^2 \pi_\delta} - \frac{\pi_\tau + \pi_\delta}{1 + \sigma^2(\pi_\tau + \pi_\delta)} \\ &= \frac{\sigma^2 \pi_\tau \pi_\delta (2 + \sigma^2 \pi_\tau + \sigma^2 \pi_\delta)}{(1 + \sigma^2 \pi_\tau)(1 + \sigma^2 \pi_\delta)[1 + \sigma^2(\pi_\tau + \pi_\delta)]} > 0, \end{aligned}$$

as stated. ∎

Proof of Corollary 1. First, note that it follows directly from the result in Proposition 1 that $\overline{\omega}_g > w([0,1])$ for any network function g. From the result shown in Lemma 1, we know that $w([0,1]) = \sum_{j=1}^{n} \pi_j / \left(1 + \sigma^2 \sum_{j=1}^{n} \pi_j\right)$.

Secondly, from the result obtained in Lemma 1, it follows that each $w(\upsilon)$ converges to $1/\sigma^2$ as the signals increase arbitrarily their precision, that is, when all π_j tend to infinity. Therefore, for any network function g, the term $\overline{\omega}_g$ cannot exceed γ_g/σ^2. In addition, since $\sum_{\upsilon \in \Upsilon_g} \mathrm{Var}_\upsilon[\theta] = \sigma^2 (\gamma_g - \overline{\omega}_g) > 0$, we must have also $\gamma_g > \overline{\omega}_g$. By combining both requirements, we obtain that $\overline{\omega}_g$ cannot exceed $\gamma_g \min\{1, 1/\sigma^2\}$, as stated. ∎

Proof of Proposition 2. From the optimal network design problem in 12, we observe that

$$\frac{\partial L(g)}{\partial \underline{\omega}_g} = \frac{(1-\lambda)^2 \sigma^2}{(1 - \lambda \underline{\omega}_g)^4} \left[1 - \lambda \underline{\omega}_g\right] \left[(2\gamma_g - \underline{\omega}_g)\lambda - 1\right].$$

Therefore, it follows that

$$\begin{aligned} \frac{\partial L(g)}{\partial \underline{\omega}_g} > 0 \;\Leftrightarrow\; & \text{ either A: } \lambda < 1/\underline{\omega}_g \text{ and } \lambda > 1/(2\gamma_g - \underline{\omega}_g) \\ & \text{or B: } \lambda < 1/\underline{\omega}_g \text{ and } \lambda > 1/(2\gamma_g - \underline{\omega}_g); \\ \frac{\partial L(g)}{\partial \underline{\omega}_g} < 0 \;\Leftrightarrow\; & \text{ either C: } \lambda > 1/\underline{\omega}_g \text{ and } \lambda > 1/(2\gamma_g - \underline{\omega}_g) \\ & \text{or D: } \lambda < 1/\underline{\omega}_g \text{ and } \lambda < 1/(2\gamma_g - \underline{\omega}_g). \end{aligned}$$

We proceed with the proof by studying each of the conditions stated in A–D above.

A: In this case, we have $\lambda \in \big(1/(2\gamma_g - \underline{\omega}_g), 1/\underline{\omega}_g\big)$, which is a well-specified interval under the restrictions that $\underline{\omega}_g$ and γ_g must satisfy. Since the loss function $L(g)$ is always increasing in this case, it follows that efficient networks are achieved when $\underline{\omega}_g = \underline{\omega}$ and the set of signals is distributed into as few neighborhoods as possible. Thus, the number of resulting neighborhoods must equal the lower bound $\underline{\gamma}$.

B: In this case we must have $\lambda \in \big(1/\underline{\omega}_g, 1/(2\gamma_g - \underline{\omega}_g)\big)$. Yet, this is not a well-specified interval under the requirement $\underline{\omega}_g \geq \gamma_g$.

C: In this case, we have $\lambda < \min\big\{1/\underline{\omega}_g, 1/(2\gamma_g - \underline{\omega}_g)\big\} = 1/(2\gamma_g - \underline{\omega}_g)$. Since the loss function $L(g)$ is always decreasing in this case, it follows that efficient networks are achieved when $\underline{\omega}_g = \overline{\omega}$ and the set of signals is distributed into as many neighborhoods as possible. Under the restriction that the network is minimally connected, this can only be the case for the connected line network. For this network, we have $\gamma_g = n$.

D: In this case, we have $\lambda > \max\big\{1/\underline{\omega}_g, 1/(2\gamma_g - \underline{\omega}_g)\big\} = 1/\underline{\omega}_g$. Since the loss function $L(g)$ is always decreasing in this case, it follows that efficient networks are achieved when $\underline{\omega}_g = \overline{\omega}$ and the set of signals is distributed into as many neighborhoods as possible. As in case C above, under the restriction that the network is minimally connected, this can only be the case for the connected line network. For this network, we have $\gamma_g = n$.

Then, the results stated in (i) and (ii) of the proposition follow directly by combining the requirements and implications stated in A–D above. ∎

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