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ANTONIO JIMÉNEZ MARTÍNEZ

Consensus in Communication Networks under Bayesian Updating

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Abstract

This paper studies the evolution of beliefs of a group of Bayesian updaters who are connected through a social network that enables them to listen to the opinions of others. Each agent observes a sequence of private signals about the value of an unknown parameter. In addition, the agent receives private messages from others according to her connections in the network. A message conveys some information about the signal observed by the sender. Both signals and messages are independent and identically distributed across time but not necessarily across agents. Messages cannot be transmitted through indirect connections in the network. We first characterize the long-run behavior of an agent's beliefs in terms of the relative entropies of the conditional distributions of signals and messages available to the agent. Then, under some mild assumptions on the distributions of signals and messages, we identify a condition under which the agents reach a consensus in their opinions even when they begin with different priors. Finally, in contrast with most results in the existing literature, we show that a consensus need not be reached in a strongly connected network.

Resumen

Este artículo estudia la evolución de las creencias de un grupo de agentes Bayesianos que están conectados a través de una red social, que les permite escuchar las opiniones de otros. Cada agente observa una secuencia de señales privadas sobre el valor de un parámetro desconocido. Además, el agente recibe mensajes privados de otros agentes según sus conexiones en la red. Un mensaje contiene información sobre la señal observada por el emisor. Tanto señales como mensajes son independientes e idénticamente distribuidos a lo largo del tiempo, pero no necesariamente entre agentes. Los mensajes no se pueden transmitir a través de conexiones indirectas en la red. Primero caracterizamos el comportamiento de largo plazo de las creencias de un agente en términos de las entropías relativas de las distribuciones condicionales de señales y mensajes disponibles al agente. Entonces, bajo algunos supuestos leves sobre las distribuciones de señales y mensajes, identificamos una condición bajo la cual los agentes alcanzan un consenso en sus opiniones incluso cuando comienzan con creencias distintas. Finalmente, a diferencia de la mayoría de resultados disponibles en la literatura, mostramos que un consenso puede no alcanzarse en una red fuertemente conectada.

Consensus in Communication Networks under Bayesian Updating

Antonio Jiménez-Martínez*

April 2012

Abstract

This paper studies the evolution of beliefs of a group of Bayesian updaters who are connected through a social network that enables them to listen to the opinions of others. Each agent observes a sequence of private signals about the value of an unknown parameter. In addition, the agent receives private messages from others according to her connections in the network. A message conveys some information about the signal observed by the sender. Both signals and messages are independent and identically distributed across time but not necessarily across agents. Messages cannot be transmitted through indirect connections in the network. We first characterize the long-run behavior of an agent's beliefs in terms of the relative entropies of the conditional distributions of signals and messages available to the agent. Then, under some mild assumptions on the distributions of signals and messages, we identify a condition under which the agents reach a consensus in their opinions even when they begin with different priors. Finally, in contrast with most results in the existing literature, we show that a consensus need not be reached in a strongly connected network.

Keywords: Communication networks, Bayesian updating, private signals, private messages, consensus.

JEL Classification: C72, D82, D83.

*División de Economía, Centro de Investigación y Docencia Económicas.

1 Introduction

Most individual and social decisions rely on the beliefs that agents form about economic and political variables based on the information they receive from neighbors, friends, coworkers, local leaders, and political actors. Social networks are primary channels that carry news, information, and opinions about products, job vacancies, and political programs. Coordinating decisions when payoffs depend on an unknown underlying parameter requires agents to have beliefs that are not too different.

Suppose that a group of agents start with different priors, that each agent observes over time a sequence of private signals about the parameter, and that, in addition, she receives, according to her connections in the network, some information about the private signals that others observe. Under which conditions on the network structure will the agents eventually have similar beliefs about the parameter value? To answer this question, we develop a tractable benchmark to study the dynamics of belief formation when agents receive private information about the parameter from an external source and, at the same time, there is communication between connected agents. We choose the notion of *consensus of beliefs*, defined as obtaining common limiting beliefs for all agents, as our criterion of what constitutes similar beliefs. We remark that this notion of consensus does not rely on any ex-ante probability assessments over histories leading to common limiting beliefs.

To illustrate the importance of having similar beliefs in coordinating decisions and making efficient outcomes possible, consider the following example, based on an example from Martin W. Cripps, Jeffrey C. Ely, George J. Mailath, and Larry Samuelson (2008). Suppose that two agents, $i = 1, 2$, play the following investment game. First, nature chooses a parameter $\theta \in \{\theta_a, \theta_b\}$ and each agent i assigns a prior probability $q_i \in [0, 1]$ to θ_a being the true parameter value. After that, in each period $t = 0, 1, 2, \ldots$, each agent observes privately a signal containing some information about parameter θ and then chooses either action A or action B. Simultaneous choices of A when the parameter is θ_a, or B when it is θ_b, gives a payoff of 1 to each. Lone choices of A or B breaks joint investments opportunities and gives a payoff of $-c$, for some $c > 0$. Joint choices that do not match the parameter gives the agents a zero payoff when the parameter is θ_a and a payoff of $-2c$ when the parameter if θ_b. Thus, in this coordination game, each player must learn individually the appropriate course of action, without communicating with the other. Figure 1 depicts these payoffs. Under what circumstances do there exist equilibria of this investment game in which the agents coordinate their actions by choosing action A?

Notice that action A will be optimal for an agent in some period t only if the agent assigns probability at least $\frac{c}{1+2c} =: q$ to θ_a being the true parameter value. Therefore, it is interesting to know whether the priors q_i will evolve over time so as to put eventually, both of them, at least probability q to θ_a being the true parameter value. Otherwise, coordination in action A will fail.

	A	B
A	$1, 1$	$-c, -c$
B	$-c, -c$	$0, 0$

Parameter θ_a

	A	B
A	$-2c, -2c$	$-c, -c$
B	$-c, -c$	$1, 1$

Parameter θ_b

FIGURE 1.— Payoffs from a potential joint opportunity with actions A and B available to each agent.

The requirement stated above, however, is a necessary but not sufficient condition. To ensure that both agents will indeed coordinate in action A, the signal process must be such that the event that "both agents attach probability at least q to θ_a being the true value parameter" become eventually commonly known by the agents.[1] In short, successful coordination requires that the agents (at least approximately) commonly learn the parameter value. That is, agent 1 must assign sufficiently high probability not only to θ_a, but also to the event that agent 2 assigns high probability to this value, and to the event that agent 2 assigns high probability to the event that agent 1 assigns high probability to θ_a, and so on. For a setting without communication between the agents, as the one illustrated by the example above, but in which the agents begin with common priors, Cripps, Ely, Mailath, and Samuelson (2008) show that (approximate) common learning of the parameter is attained when signals are sufficiently informative and the sets of signals are finite. This result follows regardless of the pattern of correlations between the agents' signals.

Unfortunately, recent research on epistemic conditions underlying solution concepts in game theory shows that, under some circumstances, the presence of communication among the agents precludes in general common learning of an event.[2] The key feature that prevents common

[1]More precisely, if $\mathcal{A}$ denotes the set of histories at which both agents choose A, then at each history in $\mathcal{A}$, each agent must assign probability at least q to $\mathcal{A}$. This is equivalent to the statement that the set of histories $\mathcal{A}$ is q-evident according to the notion proposed by Dov Monderet and Dov Samet (1989). If, in addition, at each history in $\mathcal{A}$, each agent assigns probability at least q to θ_a being the true parameter value, then θ_a is common q-belief, which turns out to be a sufficient condition for an equilibrium in which each agent chooses A.

[2]Following an observation by Rohit Parikh and Paul Krasucki (1990), Aviad Heifetz (1996) shows that communication according to a stochastic protocol may prevent common knowledge. Using an infection argument similar to that present in the email game of Ariel Rubinstein (1989), Frédéric Koessler (2001) generalizes this result by showing that common knowledge of an event cannot be achieved when there is uncertainty regarding whether messages reach receivers. Also, Jacub Steiner and Colin Stewart (2010) show that delayed communication according to a stochastic protocol may destroy common learning.

knowledge of an event in the presence of communication is the fact that messages are often correlated across time. This is typically caused by the fact that the messages received by an agent depend on the information available to the sender at the time when the message is sent. Thus, if the probability with which a message is sent changes with the information that the sender has, then messages are not independent over time. The negative implication of the correlation over time of the information received by the agents is important. In a recent paper, Martin W. Cripps, Jeffrey C. Ely, George J. Mailath, and Larry Samuelson (2012) show that common learning can be precluded even when the private signals follow very simple time dependence patterns. This feature, however, is not present in our model because signals are independent over time and conditional distributions over messages are constant over time. Taken together, both assumptions ensure that messages are independent over time.[3] Therefore, it follows from the result of Cripps, Ely, Mailath, and Samuelson (2008), that, if the agents have common priors, and messages and signals are sufficiently informative, then (approximately) common knowledge of the parameter is obtained.

However, situations in which the agents act as Bayesian updaters and yet end up with posteriors that are not accurate (at least not completely) seems the rule rather than the exception in practice. Otherwise, several important phenomena, such as propaganda, censorship, and marketing, are difficult to rationalize. For instance, suppose that an agent receives information from some media outlet with a well-known political bias. If the agent makes apriori assessments about how strong the outlet's arguments are likely to be, then she should be swayed toward the media's view only if the arguments were stronger that expected. But, by a large numbers argument, listening to this media outlet should on average have no effect over her beliefs. This result, however, seems in contrast with causal observation.

Our research question does not focus on learning issues but on the evolution of the agents' posteriors when they use Bayesian updating rules. Accordingly, as indicated earlier, we do not incorporate into our notion of consensus the possibility that the agents make ex-ante probabilistic assessments about the histories underlying their beliefs. In particular, we allow them to hold beliefs that are not accurate in the sense that there is a positive probability of deviating substantially from the truth. In other words, no reference need be made regarding whether the agents (approximately) learn the true parameter value in the long run neither whether they ex-ante believe that their posteriors will eventually converge to the same limiting beliefs. In

[3]It is also important to note that, in our model, there is neither delayed communication nor uncertainty regarding whether messages reach receivers.

this respect, our notion of what constitutes similar beliefs is different from typical concepts of agreement (see footnote 10 for further details) used in the learning literature.[4] In particular, our research question is different, though related, to that analyzed by Cripps, Ely, Mailath, and Samuelson (2008).

By analyzing whether posteriors converge to some common limiting beliefs, we investigate the conditions under which the necessary condition identified in the previous investment example can be obtained. Our exercise is similar in spirit to the one carried out in his seminal paper on consensus by Morris H. DeGroot (1974), in which he proposes the same notion of consensus as we do so that the analysis ignores whether the agents converge to beliefs close to the truth.[5] But our approach is quite different from that pursued in a recent paper by Benjamin Golub and Matthew O. Jackson (2010) on "wise" networked societies, in which they ask under which conditions will all agents converge to hold beliefs close to the true parameter value.

Our model is as follows. A set of agents begin with imperfect (and possibly asymmetric) information on an unknown parameter. Over time, each of them receives information about the parameter from some private external source and, in addition, they communicate according to an exogenous (possibly directed) social network.

More specifically, we consider two possible parameter values. Each agent (1) observes a sequence of private signals and (2) receives a sequence of private messages from each agent to whom she has a direct (directed) link. We assume that messages cannot be transmitted through indirect connections. We allow messages to be correlated with the sender's signal so that the receiver can obtain some information about the signal that the sender observes. Thus, at a more intuitive level, the network describes conduits through which agents can listen to other agents speak about the signals that they observe. An important feature of our model is that the transmission of information between two agents is modeled using a sender-receiver protocol, which is assumed to be constant over time.

We first characterize, in Proposition 1, an agent's limiting beliefs in terms of a measure that depends on the relative entropies of the distributions of signals and messages available to her

[4]For instance, in their classical justification of the common prior assumption, Leonard J. Savage (1954, p. 48), and David Blackwell and Lester Dubins (1962) establish that Bayesian agents who observe the same sequences of sufficiently informative signals will learn individually the true parameter value, and, as a consequence, they will reach an agreement. Individual learning in this context requires that, conditioned on a parameter value, the agent assigns probability one to the event that her limiting beliefs put probability one to that parameter value. Also, Daron Acemoglu, Victor Chernouzhukov, and Muhamet Yildiz (2009) use a notion of agreement that requires that the agents assign probability one to the event that their posteriors converge to the same limiting beliefs.

[5]Using the model of DeGroot (1974), Peter M. DeMarzo, Dimitri Vayanos, and Jeffrey Zwiebel (2003) consider also the notion of consensus that we use in this paper. In addition, they analyze whether limiting beliefs are correct in the sense of being close to an aggregate of the agents' initial priors rather than to the true parameter value.

in the network. Using this result, and under some mild assumptions on the distributions of signals and messages, we identify a condition under which consensus is always attained for some open set of networks. This condition, which we label as *Inter Group Connectedness (IGC)*, requires that, upon dividing the society into two groups according to the parameter value which they consider ex-ante most likely, each agent who belongs to at least one of these groups has a directed link to some agent in the other group. In Proposition 2, we show that if a consensus is attained for a network, then such a network necessarily satisfies IGC. Conversely, we show that there exists an open set of networks, satisfying IGC, for which a consensus is attained.

Turning to the literature, a number of papers have recently built upon the canonical model proposed by DeGroot (1974) to explore the evolution of beliefs over time when agents can listen to the opinions of others according to their connections in a social network. For instance, Peter M. DeMarzo, Dimitri Vayanos, and Jeffrey Zwiebel (2003) study the beliefs finally held by the agents under the assumption that they fail to adjust for repetitions of information over time. Also, Golub and Jackson (2004) use DeGroot's model to analyze the conditions on the network structure under which the agents' beliefs converge to the true parameter value. Our paper is different from this branch of the literature because in DeGroot's model the agents are boundedly rational in the sense that they use a non-Bayesian updating rule. More precisely, an agent updates her beliefs by the rule of thumb of weighting the beliefs of the agents to whom she has a directed link. The network is then characterized by a transition matrix that describes the evolution of the agents' beliefs over time. In contrast, our paper considers Bayesian updating from sequences of informative signals and messages.

Other recent work in which the agents update their beliefs according to some reasonable rules of thumb includes Daron Acemoglu, Asuman Ozdaglar, and Ali ParandehGheibi (2010), in which the agents meet pairwise and adopt the average of their pre-meeting beliefs. They study how the presence of agents who influence the beliefs of others, but do not change their own beliefs, interferes with the spread of information along the network. Although they do not consider consensus specifically, our model allows for insights with a similar flavor since some spread of beliefs among agents with different opinions is required for consensus. In our model, consensus can be prevented when an agent does not listen enough to agents with different opinions and, at the same time, is listened by others. Such an agent would play a similar role to a "forceful" agent in the model by Acemoglu, Ozdaglar and ParandehGheibi (2010). The question of whether consensus is attained under a non-Bayesian updating rule is analyzed by Daron Acemoglu, Giacomo Como, Fabio Fagnani, and Asuman Ozdaglar (2010). They distinguish between regular agents, who update their beliefs according to the information they receive from their neighbors, and stubborn agents, who never update their beliefs. They show that consensus is never obtained when the society contains stubborn agents with different opinions. Again, this insight bears some resemblance with ours when the connections of some agent does not allow her to change her opinion in the long run, e.g., if the IGC condition is not satisfied.

Another branch of the literature on learning in social networks considers that, in addition to observing signals, the agents are able to observe their neighbors' past payoffs or past actions. An important contribution within these models of observational learning is the work of Venkatesh Bala and Sanjeev Goyal (1998), in which the agents take repeated actions and can observe their neighbors' payoffs. They obtain consensus within connected components of the network since each agent can observe whether her neighbors are earning payoffs different from her own. Daron Acemoglu, Munther A. Dahleh, Ilan Lobel, and Asuman Ozdaglar (2011) consider that agents can observe their neighbors' past actions and focus on studying asymptotic learning, defined as the convergence of the agents' actions to the right action as the social network becomes large. They provide conditions on the expansion of the network under which there is asymptotic learning when private beliefs are either bounded or unbounded.

Finally, we emphasize that existing models on communication and learning, based on Bayesian and non-Bayesian updating rules, typically lead to consensus when communication takes place over a strongly connected network (e.g., Acemoglu, Dahleh, Lobel and Ozdaglar, 2008; Bala and Goyal, 1998; DeMarzo, Vayanos and Zwiebel, 2003, Golub and Jackson, 2004; Acemoglu, Ozdaglar and ParandehGheibi, 2009). Nevertheless, a strongly connected network need not satisfy condition IGC in our model, as we show in Example 1, which could preclude consensus. This difference can be explained by the fact that, in our set up, the information contained in the messages does not flow in *any* period through indirect connections in the network. This observation is particularly important since it shows that restricting communication to flow over time only locally in the social network (i.e., within each pair of directly connected neighbors) interferes with the spread of information so as to prevent consensus. Nevertheless, even though we consider that the agents are somewhat "locally isolated" in the sense that they do not pass to others the information they receive from their direct neighbors, the IGC condition is sufficient for the spread of information among *all* agents. Thus, IGC seems important to identify influential agents in the presence of restrictions for the transmission of information through indirect connections in the network.

The rest of the paper is organized as follows. Section 2 presents the model, Section 3 characterizes consensus, and Section 4 concludes. All the proofs are in the Appendix.

2 The Model

For any set Q, $\Delta(Q)$ denotes the set of all Borel probability measures on Q.

2.1 Information Processes

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. There is a finite set of agents $N := \{1, 2, \ldots, n\}$, with $n \geq 3$, who receive information from others according to an exogenously given (and possibly directed) social network. The agents care about a parameter $\theta \in \Theta := \{\theta_a, \theta_b\}$, which is selected by nature before period zero. Each agent i has a (subjective) prior distribution $p_i \in \Delta(\Theta)$ that describes her ex-ante beliefs about the parameter. The realized parameter θ is not observed directly by any agent. Instead, in each period t, each agent i observes *privately* a *signal realization* $s_{it} \in S := \{s_a, s_b\}$ and receives *privately* a *message* $m_{ikt} \in M := \{m_a, m_b\}$ from each agent $k \in N$. As it will be explained below, the message m_{ikt} only conveys information to agent i about the signal s_{kt} observed by agent k. In principle, we allow each agent to receive messages from herself as well. However, since an agent already observes her own signals, the information conveyed by such messages is redundant. Also, even though an agent receives messages from all agents, the social network restricts the amount of information that she receives from such messages. The constraints that the network imposes on the information that the agents receive from others are described more precisely in the next subsection. A message vector received by agent i in period t is denoted by $m_{it} = (m_{ikt})_{k \in N} \in M^n$.

Conditional on θ, a stochastic process $\zeta^\theta := \left\{\zeta_t^\theta\right\}_{t=0}^{\infty} = \left\{(\zeta_{it}^\theta)_{i \in N}\right\}_{t=0}^{\infty}$ generates the *signal profile* $s_t = (s_{it})_{i \in N} \in S^n$ for each t and a stochastic process $\eta^\theta := \left\{\eta_t^\theta\right\}_{t=0}^{\infty} = \left\{((\eta_{ikt}^\theta)_{k \in N_i})_{i \in N}\right\}_{t=0}^{\infty}$ generates the *message profile* $m_t = (m_{it})_{i \in N} \in M^{n^2}$ for each t. For each $\theta \in \Theta$, the process ζ^θ is independent and identically distributed across t.

For each period t, the joint conditional distribution over signal-message profiles is denoted by $\pi^\theta(s, m) := P(\zeta_t^\theta = s, \eta_t^\theta = m)$, the marginal conditional distribution over the signals observed by agent i is denoted by $\phi_i^\theta(s_i) := P(\zeta_{it}^\theta = s_i)$, and the marginal conditional distribution over the messages received by agent i from her neighbor k is denoted by $\psi_{ik}^\theta(m_{ik}) := P(\eta_{ikt}^\theta = m_{ik})$. We use ζ and η, and ϕ_i and ψ_{ik}, to denote, respectively, the corresponding unconditional stochastic processes and unconditional distributions.

An important ingredient of our model is the communication protocol through which information is transmitted between directly connected agents. Each agent i receives a message from each agent $k \in N$ (that is, including herself) in each period t according to a pair of conditional distributions $\sigma_{ik}^{s_k} \in \Delta(M)$, $s_k \in S$. These distributions $\sigma_{ik}^{s_k}(m_{ik}) := P(\eta_{ikt} = m_{ik} \,|\, \zeta_{kt} = s_k)$, $s_k \in S$, are constant across t and their shapes capture the precision of the communication transmitted from agent k to agent i about the signals observed by k.[6] Thus, each agent learns about the value of the parameter θ not only by observing her own sequence of signals but also by obtaining some information about the signals that other agents observe. One way to interpret communication in this model is by considering that each agent i listens, with some fixed precision (or degree of informativeness) given by $\sigma_{ik}^{s_k}$, $s_k \in S$, to the opinions about θ that each agent $k \in N$ forms from her own private signals.

In general, since the messages received by an agent from another depend on the signal that the sender observes (and, thus, on the information that she has), messages should be correlated across time. In our model, however, since conditional distributions over messages are constant across time and signals are independent across time, messages are also independent over time. Therefore, for each $\theta \in \Theta$, the process η^θ is also independent and identically distributed across t. Furthermore, given the distributions $\sigma_{ik}^{s_k}$, $s_k \in S$, the following consistency requirement relates the process generating agent i's messages received from agent k with k's own signal process:

$$\psi_{ik}^{\theta}(m_{ik}) = \sum_{s_k \in S} \phi_k^{\theta}(s_k)\sigma_{ik}^{s_k}(m_{ik}), \quad \forall \theta \in \Theta, \quad \forall m_{ik} \in M. \tag{1}$$

For each pair of agents $i, k \in N$, we restrict attention to the class of distributions ϕ_i^θ and $\sigma_{ik}^{s_k}$ parameterized as follows. For each subscript $\alpha \in \{a, b\}$, let $\phi_i^{\theta_\alpha}(s_\alpha) = x_i \in [1/2, 1]$ and $\sigma_{ik}^{s_\alpha}(m_\alpha) = y_{ik} \in [1/2, 1]$. Thus, without loss of generality, we are assuming that signal s_a (respectively, s_b) is more likely than signal s_b (respectively, s_a) under parameter θ_a (respectively, θ_b). Analogously, we assume that agent i receives message m_a (respectively, m_b) more often than message m_b (respectively, m_a) when agent k observes signal s_a (respectively, s_b). While the restriction to conditional distributions satisfying $\phi_i^{\theta_a}(m_a) = \phi_i^{\theta_b}(m_b)$ and $\sigma_{ik}^{s_a}(m_a) = \sigma_{ik}^{s_b}(m_b)$ is not essential for our results to follow, it simplifies greatly the analysis. With these parameterizations, we can compare in a very simple and neat way the informativeness of any two distributions over signals (and of any two distributions over messages) using the classical order introduced by David Blackwell (1951, 1953). A conditional distribution over signals associated to parameter x_i is at least as informative as a distribution associated to parameter x_i' if and only if $x_i \geq x_i'$. Analogously, agent i receives from agent k at least as much information under y_{ik} than under y_{ik}'

[6] Modeling communication between pair of agents connected in a network by means of a signaling or cheap talk protocol has been recently considered, among others, by Jeanne Hagenbach and Frédéric Koessler (2010).

whenever $y_{ik} \geq y'_{ik}$. Given this characterization of the degree of informativeness of signals and messages, we will use from here onwards x_i and y_{ik} to indicate, respectively, the precision with which agent i receives information about parameter θ through her own signals and the precision with which agent i listens to agent k speak about her signals. Let $x = (x_i)_{i \in N} \in X = [1/2, 1]^n$ denote a *signal-precision profile*. By construction, this model does not consider neither strategic information revelation nor strategic attention. Information transmission is exogenously given by the network structure, as explained in more detail in the next subsection.

2.2 Communication Networks

The agents communicate, using the distributions $\sigma_{ik}^{s_k}$ ($s_k \in S$), through an exogenous social network consisting of a set of directed links between pairs of agents. If $y_{ik} = 1/2$, then agent i learns nothing about the signals received by agent k. We interpret this as agent i not listening to the opinions of agent k about parameter θ and model this situation as agent i not having a directed link to agent k. On the opposite extreme, if $y_{ik} = 1$, then agent i obtains full information about the signals received by agent k, which we model as agent i having a directed link to agent k with the highest possible intensity.[7] Thus, our parameterization of the conditional distributions over messages allows us to capture the patterns of communication among the agents through an $n \times n$ matrix $Y = [\, y_{ik} \,]$, where $y_{ik} \in [1/2, 1]$ indicates the intensity with which agent i pays attention to agent k. The interactions can be one sided, so that $y_{ik} > 1/2$ while $y_{ki} = 1/2$. We set $y_{ii} = 1$ for each $i \in N$ so that an agent transmits full information to herself. As mentioned earlier, this information is redundant since agent i already observes her own signals.

Following the terminology used by Golub and Jackson (2010), we refer to Y as the *interaction matrix*[8] describing the pattern of communication relations in the network. Information cannot be transmitted through indirect (directed) connections in the network. Given a network described by Y, let $A_i := \{k \in N \setminus \{i\} \, : \, y_{ik} > 1/2\}$ denote the set of agents to whom agent i listens.

A *directed path* in the network described by Y is a sequence $(i_1, i_2, \ldots, i_R)$ of distinct agents such that $y_{i_r i_{r+1}} > 1/2$ for each $r = 1, 2, \ldots, R-1$. We say that the network described by Y is *strongly connected* if it has a directed path from any agent to any other agent. Throughout the

[7]Note that, in the usual terminology of sender-receiver games, $y_{ik} = 1/2$ corresponds to a pooling strategy, $y_{ik} = 1$ corresponds to a completely separating strategy, and each $y_{ik} \in (1/2, 1)$ gives a partially separating strategy.

[8]However, matrix Y is an object very different from the interaction matrix used by Golub and Jackson (2010). In particular, their interaction matrix is stochastic while Y is not. Also, their interaction matrix plays the role of a transition matrix to update beliefs while this is not the case in our model. Nevertheless, there are conceptual similarities in the sense that an entry of both matrices captures the intensity with which an agent pays attention to the opinions of another agent in the society.

paper we shall consider only strongly connected networks. Let $\mathcal{Y}$ denote the set of all interaction matrices that describe strongly connected networks.

2.3 Updating and Consensus

For each period t, let $Z_i := S \times M^n$ denote the set of all signals and message vectors for agent i and let $Z := S^n \times M^{n^2}$ denote the set of all signal-message profiles. A *state* consists of a parameter and a sequence of signal-message profiles, and it is denoted by $\omega = (\theta, \{(s_t, m_t)\}_{t=0}^{\infty}) \in \Omega := \Theta \times Z^{\infty}$. When convenient, we abuse notation by writing θ for the event $\{\theta\} \times Z^{\infty} \subset \Omega$.

A *period-t history* for agent i is a string $((s_{i0}, m_{i0}), (s_{i1}, m_{i1}), \ldots, (s_{it}, m_{it})) \in Z_i^t$ and the filtration induced on Ω by agent i's histories is denoted by $\{\mathcal{H}_{it}\}_{t=0}^{\infty}$. The posterior belief of agent i about parameter θ in each period t is given by the ($\mathcal{H}_{it}$-measurable) random variable $\mu_i(\theta \,|\, \mathcal{H}_{it}) : \Omega \to [0,1]$. For each agent i and each value of the parameter θ, the sequence of random variables $\{\mu_i(\theta \,|\, \mathcal{H}_{it})\}_{t=0}^{\infty}$ is a bounded martingale,[9] which ensures that the agents' posterior beliefs converge almost surely (see, e.g., Patrick Billingsley, 1995, Theorem 35.5).

Definition 1. A *consensus is (asymptotically) reached in the society* if the posterior beliefs of all agents converge to the same value regardless of their priors, that is, if for each $i \in N$, each $p_i \in \Delta(\Theta)$, and each $\theta \in \Theta$,

$$\lim_{t \to \infty} \mu_i(\theta \,|\, \mathcal{H}_{it}) = q, \tag{2}$$

for some $q \in [0,1]$.

Note that condition (2) says nothing about whether the agents' beliefs approximate any "objective" or "true" beliefs about the parameter. Neither it imposes any conditions on probability assessments over the set of histories at which beliefs converge to common limiting beliefs. In particular, as discussed in the Introduction, it does not require that the agents' posteriors converge to a belief close the true parameter value.[10] That is, we allow that the agents' limiting beliefs be inaccurate in the sense that they put positive probability to deviating substantially from the true parameter. As mentioned in the Introduction, this is motivated by our goal of studying the evolution of beliefs rather than studying whether the agents learn or not true events.

[9]More formally, $\{\mu_i(\theta \,|\, \mathcal{H}_{it})\}_{t=0}^{\infty}$ is a bounded martingale with respect to the measure on Ω, conditional on θ, induced by the priors $(p_i)_{i \in N}$, the signal processes $(\zeta^{\theta})_{\theta \in \Theta}$, and the message processes $(\eta^{\theta})_{\theta \in \Theta}$.

[10]Typical notions of individual learning and agreement used in the learning literature can be defined, using the notation that we have introduced, as follows. Let P denote the measure on Ω induced by the priors $(p_i)_{i \in N}$, the signal processes $(\zeta^{\theta})_{\theta \in \Theta}$, and the message processes $(\eta^{\theta})_{\theta \in \Theta}$, and let P^{θ} denote the corresponding measure conditional on θ. Then, individual learning requires that $P^{\theta}\left(\{\omega \in \Omega : \lim_{t \to \infty} \mu_i(\theta \,|\, \mathcal{H}_{it}) = 1\}\right) = 1$ for each $i \in N$, and agreement requires that $P\left(\{\omega \in \Omega : \lim_{t \to \infty} \mu_i(\theta \,|\, \mathcal{H}_{it}) = q \text{ for all } i \in N\}\right) = 1$ for some $q \in [0,1]$.

3 Results

3.1 Characterizing Belief Convergence

Given a value $\theta \in \Theta$ of the parameter, we use throughout the paper θ' to denote the other value of the parameter, i.e., $\{\theta'\} := \Theta \setminus \{\theta\}$. For each agent $i \in N$, let

$$H_i^{\theta\theta'} := \sum_{s_i \in S} \phi_i^{\theta}(s_i) \log \left[\frac{\phi_i^{\theta}(s_i)}{\phi_i^{\theta'}(s_i)} \right] \geq 0 \quad \text{and} \quad G_{ik}^{\theta\theta'} := \sum_{m_{ik} \in M} \psi_{ik}^{\theta}(m_{ik}) \log \left[\frac{\psi_{ik}^{\theta}(m_{ik})}{\psi_{ik}^{\theta'}(m_{ik})} \right] \geq 0 \quad (3)$$

denote, respectively, the *relative entropy* (or Kullback-Leiber distance) of ϕ_i^{θ} with respect to $\phi_i^{\theta'}$ and the relative entropy of ψ_{ik}^{θ} with respect to $\psi_{ik}^{\theta'}$, for $k \in A_i$. Then, the function $\xi_i : \Theta \to \mathbb{R}$ defined by

$$\xi_i(\theta) := p_i(\theta) H_i^{\theta\theta'} + \sum_{k \in A_i} p_k(\theta) G_{ik}^{\theta\theta'}$$

gives us an entropy-based measure of the weight that both the signal generating process ζ_{it} and the message generating processes η_{ikt} ($k \in A_i$) place on agent i's belief that the true state is θ (instead of θ'). The following proposition provides a characterization of the convergence of agent i's posteriors in terms of the weighted function of entropies ξ_i. This result is fairly general and does not depend on our parameterization of the distributions ϕ_i^{θ} and $\sigma_{ik}^{s_i}$.

Proposition 1. *For each sequence* $\{\mathcal{H}_{it}\}_{t=0}^{\infty}$ *of filtrations, the convergence of agent i's posteriors is characterized by (i)* $\lim_{t\to\infty} \mu_i(\theta \,|\, \mathcal{H}_{it}) = 1$ *if and only if* θ *is the unique element of* $\text{argmax}_{\theta \in \Theta} \xi_i$*, and (ii)* $\lim_{t\to\infty} \mu_i(\theta \,|\, \mathcal{H}_{it}) = p_i(\theta)$ *if and only if* $\text{argmax}_{\theta \in \Theta} \xi_i = \Theta$.

Suppose that an agent $i \in N$ were restricted to using solely the signals that she observes to update her beliefs (e.g., because she does not listen to anyone in the network, $A_i = \emptyset$). Then, for a given value $\theta \in \Theta$ of the parameter, application of Proposition 1 gives

$$\lim_{t\to\infty} \mu_i(\theta \,|\, \mathcal{H}_{it}) = 1 \Leftrightarrow \log\left(\frac{x_i}{1-x_i}\right)[2x_i - 1][1 - 2p_i(\theta)] < 0 \Leftrightarrow p_i(\theta) > p_i(\theta').$$

Thus, when an agent receives no information from the others, her long run beliefs are completely determined by her own prior beliefs. Our goal in this paper is to study how the fact that agents with different priors communicate through a strongly connected social network, specified by an interaction matrix $Y \in \mathcal{Y}$, affects the evolution of their posteriors and the achievement of a consensus.

For each agent $i \in N$ and the value θ_a of the parameter (chosen without loss of generality), we set $v_i := 1 - 2p_i(\theta_a)$ so that $v_i \leq 0$ (respectively, $v_i \geq 0$) indicates that agent i believes ex-ante that θ_a (respectively, θ_b) is a value at least as likely as θ_b (respectively, θ_a) for parameter θ. Then, we can specify the priors of the society using the vector $v = (v_i)_{i \in N} \in [-1, 1]^n$.

Let $N_a := \{i \in N : -1 \leq v_i \leq 0\}$ denote the set of agents who believe ex-ante that θ_a is at least as likely as θ_b and let $N_b := \{i \in N : 0 \leq v_i \leq 1\}$ denote the set of agents who believe ex-ante that θ_b is at least as likely as θ_a. Thus, we split the society into two sets of agents, those who are ex-ante in favor of θ_a (N_a) and those who are ex-ante in favor of θ_b (N_b). We assume that the agents' priors are uniformly bounded away from the distributions that assign probability one to either each of the two parameter values, and from the distributions that assign the same probability (1/2) to both parameter values as well. Specifically, we assume that there exist two intervals $V_a = [\underline{v}_a, \overline{v}_a]$ and $V_b = [\underline{v}_b, \overline{v}_b]$, with $-1 < \underline{v}_a < \overline{v}_a < 0 < \underline{v}_b < \overline{v}_b < 1$, such that $v_i \in V_a$ for each $i \in N_a$ and $v_i \in V_b$ for each $i \in N_b$. Also, to make the problem interesting, we impose that $N_a \neq \emptyset$ and $N_b \neq \emptyset$ so that there is some heterogeneity in the agents' priors.

For a pair of agents $i, k \in N$, we define

$$\delta_i := \log \left[\frac{x_i}{1 - x_i} \right] [2x_i - 1] \geq 0 \tag{4}$$

and

$$\rho_{ik} := \log \left[\frac{x_k y_{ik} + (1 - x_k)(1 - y_{ik})}{x_k(1 - y_{ik}) + (1 - x_k) y_{ik}} \right] [2x_k - 1][2y_{ik} - 1] \geq 0. \tag{5}$$

Under our parameterization of ϕ_i^θ, it follows from Proposition 1 that the product $\delta_i v_i$ measures the speed of convergence for agent i's beliefs which is due solely to the information that she receives from her own signals. Also, under our parameterization of $\sigma_{ik}^{s_k}$ and ϕ_k^θ, Proposition 1, together with the consistency requirement in (1), implies that the speed of convergence of the sequence $\{\mu_i(\theta_a \,|\, \mathcal{H}_{it})\}_{t=0}^{\infty}$, which is due solely to the information that agent i receives from agent k, is given by the product $\rho_{ik} v_k$. The following corollary follows then from Proposition 1 under our parameterization of the distributions ϕ_i^θ and $\sigma_{ik}^{s_i}$.

Corollary 1. *A consensus in which all agents believe in the long run with probability one that the value of parameter θ is θ_a (respectively, θ_b) is reached in the society if and only if for each $i \in N$, and for each $v \in [-1, 1]^n$ such that $v_j \in V_a$ for $j \in N_a$ and $v_j \in V_b$ for $j \in N_b$,*

$$\delta_i v_i + \sum_{k \in A_i \cap N_a} \rho_{ik} v_k + \sum_{k \in A_i \cap N_b} \rho_{ik} v_k < 0 \quad (\textit{respectively, } > 0), \tag{6}$$

where δ_i and each ρ_{ik} are as defined in (4) and (5).

The result is intuitive. The long run behavior of an agent's beliefs is affected by the influence of all agents to whom she listens in the network, those whose ex-ante opinion she agrees on as well as those with a different ex-ante opinion. Then, the final effect that determines the long run behavior of her beliefs is given by the expression in (6), which turns out to be linear in the agents' priors.

3.2 A Necessary (and Almost Sufficient) Condition for Consensus

From condition (6), we observe that the achievement of consensus is closely related to the fact that some agents can be influenced by agents with *different* priors. Roughly speaking, with heterogeneous priors, consensus requires that some agents change their minds over time by listening to other agents. Thus, a key condition for consensus is that each of the agents who favor ex-ante at least one given parameter value has a directed link to at least one agent who favors ex-ante the other parameter value. We label this condition as *Inter Group Connectedness (IGC)*. Specifically,

ICG: *either each agent from N_a has a directed link to at least one agent from N_b, or each agent from N_b has a directed link to at least one agent from N_a, or both.*

Proposition 2. *Suppose that a consensus is reached in the society for a network described by some interaction matrix $Y \in \mathcal{Y}$, then such a network satisfies IGC. Moreover, there exists an open set $X' \subset X$ of signal-precision profiles and an open set of interaction matrices $\mathcal{Y}' \subset \mathcal{Y}$, satisfying IGC, such that a consensus in the society is reached for each $(x, Y) \in X' \times \mathcal{Y}'$.*

It turns out that requiring that the network be strongly connected does not suffice for IGC, as the following example illustrates.

Example 1. Suppose that $n = 4$, that $N_a = \{1, 2\}$ and $N_b = \{3, 4\}$, and consider the network depicted in Figure 2.

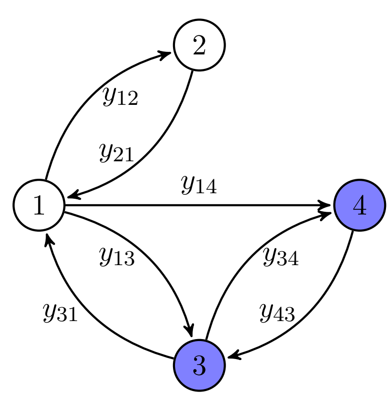

FIGURE 2.— A strongly connected network which does not satisfy IGC.

The corresponding interaction matrix is

$$Y = \begin{pmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{21} & 1 & 1/2 & 1/2 \\ y_{31} & 1/2 & 1 & y_{34} \\ 1/2 & 1/2 & y_{34} & 1 \end{pmatrix},$$

where $y_{12}, y_{13}, y_{14}, y_{21}, y_{31}, y_{34}, y_{34} \in (1/2, 1]$. Notice that agent 1 has a direct link to each other agent. Agent 2 has a direct link to agent 1 and indirect connections (via agent 1) to agents 3 and 4. Also, agent 3 has direct links to agents 1 and 4, and an indirect connection (via agent 1) to agent 2. Finally, agent 4 has a direct link to agent 3, an indirect connection (via agent 3) to agent 1, and another indirect connection (via agents 3 and 1) to agent 2. Therefore, each agent has a directed path to each other agent so that the network is strongly connected. However, this network does not satisfy IGC since agent $2 \in N_a$ has no directed link to any agent in N_b and agent $4 \in N_b$ has no directed link to any agent in N_a. If follows from Corollary 1 that the limiting beliefs of agent 2 will put probability one to θ_a while the limiting beliefs of agent 4 will assign probability one to θ_b so that consensus is not attained.

4 Concluding Comments

We have assumed that there are only two possible values of the parameter. The intuition underlying the results in Propositions 1 and 2 is compelling and general. With finitely many parameter values, an agent's limiting beliefs will depend, in a way totally analogous to that stated in Corollary 1, on how she updates her beliefs by observing her own signals and by receiving messages from others according to her network connections. Also, a connectivity condition, analogous to IGC will be essential to determine whether or not consensus is attained. That condition must require that each agent in the society has a directed link to at least one agent who belongs to a group who ex-ante believes that at least one parameter value is the most likely one. In other words, for at least one pre-specified parameter value, the connections in the network must ensure that each agent may eventually change her mind by listening to others so as to put, in the long run, probability one to that parameter value being the true one. This group of agents will thus be a highly influential one.

As we discussed in the Introduction, we are ultimately interested in studying the evolution of the agents' beliefs without analyzing whether they eventually learn the truth. An interesting open question that remains concerns whether the agents ex-ante believe that they will eventually reach a consensus, without necessarily learning the truth. We believe that our model could be useful to address this question, perhaps by following an approach similar to that pursued by Acemoglu, Chernozhukov, and Yildiz (2009). Another interesting extension of the model would be that of endogenizing the listening structure. To follow this approach, more structure should be added to the model so as to consider that the agents pursue the maximization of a payoff that depends on the unknown parameter. Then, by characterizing listening structures that are "stable," one should obtain some insights into the formation of communication networks in a dynamic framework of belief evolution.

Finally, the assumption that messages are not transmitted through indirect connections seems more realistic in some environments than in others. For example, it is a natural assumption in networks within formal organizations, in which, by regulation, information is only transmitted between directly connected parts of the organization, or in networks in which there are physical restrictions to the flow of indirect information. It seems less compelling, however, in informal networks. It would be interesting to analyze the achievement of consensus if this assumption is relaxed and some amount of information is allowed to flow through indirect connections.

Appendix

Proof of Proposition 1. Given a sequence of filtrations $\{\mathcal{H}_{it}\}_{t=0}^{\infty}$, let $\alpha^{\mathcal{H}_{it}}(s_i) := |\{\tau \leq t \,:\, \zeta_{it} = s_i\}|$ be the number of periods in which agent i has observed signal s_i before period t and let $\beta_k^{\mathcal{H}_{it}}(m_{ik}) := |\{\tau \leq t \,:\, \eta_{ikt} = m_{ik}\}|$ be the number of periods in which agent i has received message m_{ik} from agent k before period t. Fix a sequence $\{\mathcal{H}_{it}\}_{t=0}^{\infty}$. Application of Bayes rule gives

$$\mu_i(\theta\,|\,\mathcal{H}_{it}) = \left[1 + \frac{p_i(\theta')}{p_i(\theta)} \prod_{s_i} \left(\frac{\phi_i^{\theta'}(s_i)}{\phi_i^{\theta}(s_i)}\right)^{\alpha^{\mathcal{H}_{it}}(s_i)} \prod_{k\in A_i}\prod_{m_{ik}} \left(\frac{\psi_{ik}^{\theta'}(m_{ik})}{\psi_{ik}^{\theta}(m_{ik})}\right)^{\beta_k^{\mathcal{H}_{it}}(m_{ik})}\right]^{-1}.$$

Since observed frequencies approximate distributions, i.e., $\lim_{t\to\infty}\alpha^{\mathcal{H}_{it}}(s_i) = \lim_{t\to\infty}[t\,\phi_i(s_i)]$ and $\lim_{t\to\infty}\beta_k^{\mathcal{H}_{it}}(m_{ik}) = \lim_{t\to\infty}[t\,\psi_{ik}(m_{ik})]$, we have

$$\lim_{t\to\infty}\mu_i(\theta\,|\,\mathcal{H}_{it}) = \left[1 + \frac{p_i(\theta')}{p_i(\theta)}\left[\prod_{s_i}\left(\frac{\phi_i^{\theta'}(s_i)}{\phi_i^{\theta}(s_i)}\right)^{\phi_i(s_i)} \prod_{k\in A_i}\prod_{m_{ik}}\left(\frac{\psi_{ik}^{\theta'}(m_{ik})}{\psi_{ik}^{\theta}(m_{ik})}\right)^{\psi_{ik}(m_{ik})}\right]^{\infty}\right]^{-1}.$$

Therefore, studying the converge of $\mu_i(\theta\,|\,\mathcal{H}_{it})$ reduces to studying whether the term

$$\prod_{s_i}\left[\frac{\phi_i^{\theta'}(s_i)}{\phi_i^{\theta}(s_i)}\right]^{\phi_i(s_i)} \prod_{k\in A_i}\prod_{m_{ik}}\left[\frac{\psi_{ik}^{\theta'}(m_{ik})}{\psi_{ik}^{\theta}(m_{ik})}\right]^{\psi_{ik}(m_{ik})}$$

exceeds or not 1. By taking logs, this is equivalent to studying whether

$$\sum_{s_i}\phi_i(s_i)\log\left[\frac{\phi_i^{\theta'}(s_i)}{\phi_i^{\theta}(s_i)}\right] + \sum_{k\in A_i}\sum_{m_{ik}}\psi_{ik}(m_{ik})\log\left[\frac{\psi_{ik}^{\theta'}(m_{ik})}{\psi_{ik}^{\theta}(m_{ik})}\right]$$

exceeds or not zero. Then, using the fact that $\phi_i(s_i) = \sum_\theta p_i(\theta)\phi_i^\theta(s_i)$ and $\psi_{ik}(m_{ik}) = \sum_\theta p_k(\theta)\psi_{ik}^\theta(m_{ik})$, together with the definitions of relative entropies in (3), we obtain that

$$\lim_{t\to\infty}\mu_i(\theta\,|\,\mathcal{H}_{it}) = 1 \;\Leftrightarrow\; p_i(\theta)H_i^{\theta\theta'} + \sum_{k\in A_i} p_k(\theta)G_{ik}^{\theta\theta'} > p_i(\theta')H_i^{\theta'\theta} + \sum_{k\in A_i} p_k(\theta')G_{ik}^{\theta'\theta}, \qquad \text{(i)}$$

and

$$\lim_{t\to\infty}\mu_i(\theta\,|\,\mathcal{H}_{it}) = p(\theta) \;\Leftrightarrow\; p_i(\theta)H_i^{\theta\theta'} + \sum_{k\in A_i} p_k(\theta)G_{ik}^{\theta\theta'} = p_i(\theta')H_i^{\theta'\theta} + \sum_{k\in A_i} p_k(\theta')G_{ik}^{\theta'\theta}, \qquad \text{(ii)}$$

as stated. ∎

Proof of Proposition 2. The first part follows trivially from Corollary 1. Suppose that the network described by $Y \in \mathcal{Y}$ does not satisfy IGC. Then, some agent $i \in N_a$ does not have a link to any agent from N_b. Therefore, $A_i \cap N_b = \emptyset$ and $\delta_i v_i + \sum_{k \in A_i \cap N_a} \rho_{ik} v_k < 0$ for each $v \in [-1, 1]^n$ such that $v_j \in V_a$ for $j \in N_a$ and $v_j \in V_b$ for $j \in N_b$. It follows from Corollary 1 that $\lim_{t \to \infty} \mu_i(\theta_a \,|\, \mathcal{H}_{it}) = 1$ for each sequence $\{\mathcal{H}_{it}\}_{t=0}^{\infty}$. In addition, some agent $j \in N_b$ does not have a link to any other agent from N_a. Then, by an analogous argument, we have $\lim_{t \to \infty} \mu_j(\theta_b \,|\, \mathcal{H}_{jt}) = 1$ for each sequence $\{\mathcal{H}_{jt}\}_{t=0}^{\infty}$, so that condition (2) cannot be satisfied and a consensus in the society is not reached.

The key to prove the second part is to show that, under IGC, we can always find an open set of interaction matrices such that the limiting beliefs of all agents who favor ex-ante a certain parameter value θ put probability one to the other parameter value θ'. Consider a network described by some interaction matrix $Y \in \mathcal{Y}$ and suppose that it satisfies IGC. Suppose without loss of generality that $A_i \cap N_a \neq \emptyset$ for each $i \in N_b$.

If there is some agent $j \in N_a$ such that $A_j \cap N_b = \emptyset$, then it follows from Corollary 1 that $\lim_{t \to \infty} \mu_j(\theta_a \,|\, \mathcal{H}_{jt}) = 1$ for each sequence $\{\mathcal{H}_{jt}\}_{t=0}^{\infty}$. In this case, a consensus in the society is reached only if each agent's limiting beliefs put probability one to θ_a. Thus, we proceed by showing that the interaction matrix Y can be chosen from an open subset of $\mathcal{Y}$ in a way such that $\lim_{t \to \infty} \mu_i(\theta_a \,|\, \mathcal{H}_{it}) = 1$ for each sequence $\{\mathcal{H}_{it}\}_{t=0}^{\infty}$ and for each $i \in N$. For each agent $i \in N_b$, consider the function $F_{ib} : X \times \mathcal{Y} \to \mathbb{R}$ specified as

$$F_{ib}(x, Y) := \sum_{k \in A_i \cap N_a} \rho_{ik} + \left(\frac{\overline{v}_b}{\overline{v}_a} \right) \left[\delta_i + \sum_{k \in A_i \cap N_b} \rho_{ik} \right],$$

where δ_i and each ρ_{ik} are as defined in (4) and (5). It can be verified that

$$\delta_i v_i + \sum_{k \in A_i \cap N_a} \rho_{ik} v_k + \sum_{k \in A_i \cap N_b} \rho_{ik} v_k < 0$$

for each vector of priors $v \in [-1, 1]^n$ satisfying $v_j \in V_a$ for $j \in N_a$ and $v_j \in V_b$ for $j \in N_b$ if and only if $F_{ib}(x, Y) > 0$. Since $A_i \cap N_a \neq \emptyset$, we can pick some $(\overline{x}, \overline{Y}) \in X \times \mathcal{Y}$ such that $F_{ib}(\overline{x}, \overline{Y}) > 0$. Given the form of the functions in (4) and (5), we observe that this is achieved by (a) choosing $\overline{x}_i$ sufficiently close to 1/2 and, for each $k \in A_i \cap N_b$, either $\overline{x}_k$ or $\overline{y}_{ik}$ sufficiently close to 1/2, and (b) choosing $\overline{y}_{ik}$ sufficiently close to one for some $k \in A_i \cap N_a$. Furthermore, from (4), (5), and (6), we see that F_{ib} is continuous, with respect to the Euclidean distance, for each (x, Y) in the interior of $X \times \mathcal{Y}$. Therefore, there is an open set $X_{ib} \times \mathcal{Y}_{ib} \subset X \times \mathcal{Y}$ such that for each $F_{ib}(x, Y) > 0$ for each $(x, Y) \in X_{ib} \times \mathcal{Y}_{ib}$.

Now, for each agent $i \in N_a$, consider the function $F_{ia} : X \times \mathcal{Y} \to \mathbb{R}$ specified as

$$F_{ia}(x, Y) := \sum_{k \in A_i \cap N_b} \rho_{ik} + \left(\frac{\overline{v}_a}{\underline{v}_b} \right) \left[\delta_i + \sum_{k \in A_i \cap N_a} \rho_{ik} \right],$$

where δ_i and each ρ_{ik} are as defined in (4) and (5). It can be verified that

$$\delta_i v_i + \sum_{k \in A_i \cap N_a} \rho_{ik} v_k + \sum_{k \in A_i \cap N_b} \rho_{ik} v_k < 0$$

for each vector of priors $v \in [-1,1]^n$ satisfying $v_j \in V_a$ for $j \in N_a$ and $v_j \in V_b$ for $j \in N_b$ if and only if $F_{ia}(x, Y) < 0$. We can pick some $(\overline{x}, \overline{Y}) \in X \times \mathcal{Y}$ such that $F_{ia}(\overline{x}, \overline{Y}) < 0$. Given the form of the functions in (4) and (5), this is achieved by (a) either choosing $\overline{x}_i$ sufficiently close to one or choosing $\overline{x}_k$ and $\overline{y}_{ik}$, for some $k \in A_i \cap N_a$, sufficiently close to one, and (b) choosing $\overline{y}_{ik}$ sufficiently close to $1/2$ for each $k \in A_i \cap N_b$. Furthermore, from (4), (5), and (6), we see that F_{ia} is continuous, with respect to the Euclidean distance, for each (x, Y) in the interior of $X \times \mathcal{Y}$. Therefore, there is an open set $X_{ia} \times \mathcal{Y}_{ia} \subset X \times \mathcal{Y}$ such that for each $F_{ia}(x, Y) < 0$ for each $(x, Y) \in X_{ia} \times \mathcal{Y}_{ia}$. Then, by choosing $X' \times \mathcal{Y}' := \bigcap_{\alpha \in \{a,b\}} \bigcap_{i \in N_\alpha} X_{i\alpha} \times \mathcal{Y}_{i\alpha}$, we obtain a consensus in the society for each $(x, Y) \in X' \times \mathcal{Y}'$ in which all agents believe in the long run that θ_a is the true parameter value.

If $A_j \cap N_b \neq \emptyset$ for each agent $j \in N_a$ (that is, each agent from N_a has a directed link to at least one agent from N_b and each agent from N_b has a directed link to at least one agent from N_a), then the previous arguments can be easily adapted, with minor qualifications, to show that there is an open and sense subset of $X \times \mathcal{Y}$ such that all agents' limiting beliefs put probability one to a particular pre-specified parameter value. All we needed to do is first pick a particular θ as the parameter value which will be chosen with probability one under each agent's limiting beliefs. Then, we simply need to repeat the previous method with arguments totally analogous to those used above. ∎

References

[1] Acemoglu, D., Bimpikis, K., and A. Ozdaglar (2011): "Dynamics of Information Exchange in Endogenous Social Networks," mimeo.

[2] Acemoglu, D., Chernozhukov, V., and M. Yildiz (2009): "Fragility of Asymptotic Agreement under Bayesian Learning," mimeo.

[3] Acemoglu, D., Como, D., Fagnani, D., and A. Ozdaglar (2010): "Opinion Fluctuations and Disagreement in Social Networks," mimeo.

[4] Acemoglu, D., Dahleh, M. A., Lobel, I., and A. Ozdaglar (2011): "Bayesian Learning in Social Networks," *Review of Economic Studies*, forthcoming.

[5] Acemoglu, D., Ozdaglar, O., and A. ParandehGheibi (2010): "Spread of (Mis)information in Social Networks," *Games and Economic Behavior*, 70, 194-227.

[6] Bala, V. and S. Goyal (1998): "Learning from Neighbors," *Review of Economic Studies*, 65, 595-621.

[7] Blackwell, D. (1951): "Comparisons of Experiments," *Proceedings in the Second Berkeley Symposium of Mathematical Statistics*, 93-102.

[8] Blackwell, D. (1953): "Equivalent Comparisons of Experiments," *Annals of Mathematical Statistics*, 24, 265-272.

[9] Blackwell, D. and L. Dubins (1962): "Merging of Opinions with Increasing Information," *Annals of Mathematical Statistics*, 33, 882-887.

[10] Creeps, M. W., Ely, J. C., Mailath, G. J., and L. Samuelson (2008): "Common Learning," *Econometrica*, 76, 4, 909-933.

[11] Creeps, M. W., Ely, J. C., Mailath, G. J., and L. Samuelson (2012): "Common Learning with Intertemporal Dependence," *International Journal of Game Theory*, forthcoming.

[12] Billingsley, P. (1995): *Probability and Measure* (Third Ed.), Wiley & Sons.

[13] DeGroot, M. H. (1974): "Reaching a Consensus," *Journal of the American Statistical Association*, 69, 345, 118-121.

[14] DeMarzo, P., Vayanos, D., and J. Zwiebel (2003): "Persuasion Bias, Social Influence, and Unidimensional Opinions," *Quarterly Journal of Economics*, 118:3, 909-968.

[15] Golub, B. and M. O. Jackson (2010): "Naïve Learning in Social Networks and the Wisdom of Crowds," *American Economic Journal: Microeconomics*, 2:1, 112-149.

[16] Hagenbach J. and F. Koessler (2010): "Strategic Communication Networks," *Review of Economic Studies*, 77, 3, 1072-1099.

[17] Heifetz A. (1996): "Comment on Consensus without Common Knowledge," *Journal of Economic Theory*, 70, 273-277.

[18] Koessler, F. (2001): "Common Knowledge and Consensus with Noisy Communication," *Mathematical Social Sciences*, 42, 139-159.

[19] Monderer, D., and D. Samet (1989): "Approximating Common Knowledge With Common Beliefs," *Games and Economic Behavior*, 1, 170-190.

[20] Parikh, R., and P. Krasucki (1990): "Communication, Consensus, and Knowledge," *Journal of Economic Theory*, 52, 1, 178-189.

[21] Rubinstein, A. (1989): "The Electronic Mail Game: Strategic Behavior under 'Almost Common Knowledge'," *American Economic Review*, 79, 3, 385-391.

[22] Savage, L. J. (1954): *The Foundations of Statistics*, Dover reprint, New York, 1972.

[23] Steiner, J., and C. Stewart (2010): "Communication, Timing, and Common Learning," mimeo.

División de Administración Pública

Dolores Luna *et al.*, *Índice de Gobierno Electrónico Estatal: La medición 2010*, DTAP-264

Gabriel Purón Cid y J. Ramón Gil-García, *Los efectos de las características tecnológicas en los sitios web del gobierno*, DTAP-263

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